

New classes of n -copy undistillable quantum states with negative partial transposition

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Abstract

The discovery of entangled quantum states from which one cannot distill pure entanglement constitutes a fundamental recent advance in the field of quantum information. Such bipartite bound-entangled (BE) quantum states *could* fall into two distinct categories: (1) Inseparable states with positive partial transposition (PPT), and (2) States with negative partial transposition (NPT). While the existence of PPT BE states has been confirmed, *only one* class of *conjectured* NPT BE states has been discovered so far. We provide explicit constructions of a variety of multi-copy undistillable NPT states, and conjecture that they constitute families of NPT BE states. For example, we show that for every pure state of Schmidt rank greater than or equal to three, one can construct n -copy undistillable NPT states, for any $n \geq 1$. The abundance of such conjectured NPT BE states, we believe, considerably strengthens the notion that being NPT is only a necessary condition for a state to be distillable.

In the past decade, the search for efficient tools to determine whether a given quantum state is entangled [1], and if so, whether it can be potentially used in quantum information processing protocols has led to several fundamental results about the nature of quantum entanglement [2, 3, 4, 5, 6]. Almost all quantum communication protocols, such as teleportation [7] and super-dense coding [8], require maximally entangled states that are shared among the spatially separated parties in conjunction with classical communication. However, entangled states are noisy in general, due to environment induced decoherence effects. Hence, in order for an entangled state to be useful, one should be able to extract maximally entangled states (in the asymptotic sense) starting from an ensemble of the given state, while using only local operations and classical communication (LOCC). States which allow such extraction of maximally entangled states are referred to as distillable quantum states. Generalizations of classical information theory concepts have led to protocols for distillation of quantum entanglement from certain classes of quantum states [9, 10, 11, 12].

Recent results have shown that even though most entangled states are distillable, some are not. The undistillable but entangled quantum states are said to possess bound entanglement [5, 13, 14, 15]. Bound entangled states cannot be prepared locally as they are entangled

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and being not distillable, cannot be directly used in quantum communication protocols. Interestingly, they can still enable non trivial quantum processes, such as activation [14, 16] and superactivation of entanglement [17], remote concentration of information [18], that are not possible with separable states. In this sense, the nonlocal properties of bound entanglement can be distinctly utilized.

Many of the properties of entangled and distillable states are studied using the partial transposition (PT) operation [2, 3]. Let ρ^{AB} be a density matrix corresponding to a bipartite quantum system consisting of subsystems A and B . Then the partial transposition operation in an orthogonal product basis is defined as: $(\rho_{m\mu,n\nu})^{PT} = \rho_{m\nu,n\mu}$, where the transpose is taken with respect to the subsystem B . If $\rho^{PT} \geq 0$, the state is said to be positive under PT (PPT), otherwise it is said to be NPT. If a state is not entangled (i.e., separable) then it must be PPT. For quantum systems in $2 \otimes 2$ and $2 \otimes 3$, the negativity under PT (NPT) is a necessary and sufficient for inseparability but only sufficient in higher dimensions [2, 3]. It was proved that PPT states are not distillable [21] and therefore, inseparable PPT states are bound entangled. This leaves an interesting open question: Are all NPT states distillable? If the answer is yes, then negativity under PT would be the necessary and sufficient condition for distillability. However, it turns out that even though most NPT states are distillable, some are possibly not. The existence of NPT states that are not distillable has been conjectured in Refs. [19, 20].

A state ρ is said to be distillable if and only if there exists a positive integer n and a Schmidt-rank (SR) two state $|\phi\rangle$ such that $\langle\phi|(\rho^{PT})^{\otimes n}|\phi\rangle < 0$ [21]. Intuitively this means that in the tensor product Hilbert space $H^{\otimes n}$ there exists a $2 \otimes 2$ subspace where the state is inseparable. Thus, a state is *n-copy undistillable* if this condition is not satisfied by n copies of the state. To prove that NPT bound entangled states exist, one needs to show that the *same state* is *n-copy undistillable* for all $n \geq 1$. In Refs [19, 20] the conjectured NPT bound entangled states were proved to be one copy undistillable. Moreover, for any given number of copies, n , states that are *n-copy undistillable* were constructed. We explore this issue further, and provide evidence that such conjectured NPT BE states can be found at infinitely many neighborhoods of the Hilbert space.

General Approach

We now introduce a general technique [19] to construct *n-copy undistillable* NPT states. Consider a class of bipartite $d \times d$ density matrices $\rho(\epsilon)$, $0 \leq \epsilon \leq 1$, where $\rho(\epsilon)$ is NPT when $\epsilon > 0$, and PPT for $\epsilon = 0$. Let us also assume that *the null space of the partial transpose of the density matrix $(\rho(\epsilon = 0))^{\otimes n}$, for all $d \geq 3$ and $n \geq 1$, does not contain any non-zero vector of Schmidt rank less than three.* Now consider the following function

$$f(\epsilon, n) = \min_{SR(|\phi\rangle)=2} \langle\phi|(\rho^{PT}(\epsilon))^{\otimes n}|\phi\rangle, \quad (1)$$

where, as shown, the minimum is taken over all Schmidt rank two states in the full $d^n \otimes d^n$ Hilbert space. Since by assumption any state $|\phi\rangle$ of Schmidt rank 2 does not lie in the null space of $(\rho^{\otimes n}(\epsilon=0))^{PT}$,³ $f(\epsilon=0, n) > 0$. Hence, it follows from continuity arguments that for all n there exists an ϵ_n such that $f(\epsilon_n, n) \geq 0$, when ϵ is in the interval $0 \leq \epsilon \leq \epsilon_n$. This implies that there is a finite range of $\epsilon > 0$ for every n such that $\rho(\epsilon)$ is n -copy *undistillable*. However, this argument is not sufficient to conclude complete undistillability because we have not established any result about the asymptotic behaviour of ϵ_n as $n \rightarrow \infty$. It might so happen that $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and then we cannot guarantee the existence of a state that is undistillable for any number of copies.

The purpose of the present note is to show that for quantum systems in $d_1 \otimes d_2$, $d_1, d_2 \geq 3$, one can construct several classes of states $\rho(\epsilon)$, $0 \leq \epsilon \leq 1$ such that **(1)** $\rho(\epsilon)$ is NPT when $\epsilon > 0$, and PPT for $\epsilon = 0$, and **(2)** the null space of the partial transpose of the density matrix, $(\rho(\epsilon=0))^{\otimes n}$, for any $n \geq 1$, does not contain any nonzero vector of Schmidt rank less than three. Then by arguments of the preceding paragraph, for any $n \geq 1$ we can generate states that are n -copy undistillable.

Constructions

We first provide constructions of such $\rho(\epsilon)$ for a pair of states $(\sigma, |\varphi\rangle\langle\varphi|)$ that satisfy the following properties.

Let σ be an NPT state and let $|\varphi\rangle$ be a pure entangled state of Schmidt rank k , $3 \leq k \leq \min(d_1, d_2)$, $|\varphi\rangle = \sum_{i=0}^{k-1} \lambda_i |ii\rangle$, where $\sum_{i=0}^{k-1} |\lambda_i|^2 = 1$ and λ_i 's are real and positive such that $\langle\varphi|\sigma^{PT}|\varphi\rangle = -|\Lambda|$.

We will later show that *such combinations (σ, φ) of states are easy to construct*. In fact one can start with any arbitrary $|\varphi\rangle$ and accordingly choose σ and vice versa.

Very recently it has been shown that the operator $\frac{1}{D-1}(\mathbf{I} - |\varphi\rangle\langle\varphi|)^{PT}$ where \mathbf{I} is the identity operator of the total Hilbert space H , is a separable density matrix [23]. The proof that it is PPT is based on the eigen-decomposition of the partial transposed operator $(|\varphi\rangle\langle\varphi|)^{PT}$:

$$(|\varphi\rangle\langle\varphi|)^{PT} = \sum_{i=0}^{k-1} \lambda_i^2 |ii\rangle\langle ii| + \sum_{i,j=0, i<j}^{k-1} \lambda_i \lambda_j |\psi_{ij}^+\rangle\langle\psi_{ij}^+| - \sum_{i,j=0, i<j}^{k-1} \lambda_i \lambda_j |\psi_{ij}^-\rangle\langle\psi_{ij}^-|, \quad (2)$$

where $|\psi_{ij}^\pm\rangle = \frac{1}{\sqrt{2}}(|ij\rangle \pm |ji\rangle)$.

We now construct the following density matrix

$$\rho(\epsilon) = \epsilon\sigma + \frac{(1-\epsilon)}{d^2-1}(\mathbf{I} - |\varphi\rangle\langle\varphi|)^{PT}. \quad (3)$$

³To be more precise, one needs to show that no Schmidt-Rank two vector can lie arbitrarily close to the null space of $(\rho^{\otimes n}(\epsilon=0))^{PT}$. However, it turns out that for a finite n it is sufficient to show that no Schmidt-Rank two vector lies in the null space.

It is easy to verify that the density matrix $\rho(\epsilon)$ is NPT when $\epsilon > 0$, and separable for $\epsilon = 0$: The negativity follows from $\langle \varphi | \rho^{PT}(\epsilon) | \varphi \rangle = \epsilon \langle \varphi | \sigma^{PT} | \varphi \rangle = -\epsilon |\Lambda|$.

The following result comprises the next crucial step in our proof for the existence of n -copy undistillable states.

Lemma 1 Given a ρ as defined in Eq.(3), the null space of $(\rho^{PT}(\epsilon = 0))^{\otimes n}$, for all $d \geq 3$ and $n \geq 1$ does not contain any nonzero vector of Schmidt rank less than three.

Proof: For a single copy the result is obvious since the only state that lies in the null space of $\rho^{PT}(\epsilon = 0)$ is $|\varphi\rangle$ which of course has Schmidt rank greater than two by construction. Before we outline our proof for n -copies, it is instructive to work with two copies in detail because the proof contains all the essential elements that we need for the case involving n -copies.

Let the following set be the basis for each of the Hilbert spaces concerned:

$$\left[\{|\varphi_l\rangle\}_{l=0}^{k-1}, \{ |ij\rangle \}_{i,j(i<j)=0}^{k-1}, \{ |ij\rangle \}_{i,j=k}^{d-1} \right], \quad (4)$$

where $|\varphi_l\rangle = \sum_{i=0}^{k-1} \lambda_{il} |ii\rangle$. Note that $\langle \varphi_l | \varphi_s \rangle = \delta_{ls}$ and furthermore, in this notation $|\varphi_0\rangle = |\varphi\rangle$. The following set comprises a basis for the null space of the operator $\frac{1}{(d^2-1)^2} (\mathbf{I} - |\varphi\rangle \langle \varphi|)^{\otimes 2}$:

$$\left[\begin{aligned} & |\varphi_0^1\rangle \otimes |\varphi_0^2\rangle, |\varphi_0^1\rangle \otimes \{ |\varphi_l^2\rangle \}_{l=1}^{k-1}, |\varphi_0^1\rangle \otimes \{ |ij\rangle^2 \}_{i,j(i<j)=0}^{k-1}, |\varphi_0^1\rangle \otimes \{ |ij\rangle \}_{i,j=k}^{d-1}, \\ & \{ |\varphi_l^1\rangle \}_{l=1}^{k-1} \otimes |\varphi_0^2\rangle, \{ |ij\rangle^1 \}_{i,j(i<j)=0}^{k-1} \otimes |\varphi_0^2\rangle, \{ |ij\rangle^1 \}_{i,j=k}^{d-1} \otimes |\varphi_0^2\rangle \end{aligned} \right]. \quad (5)$$

Note that the superscripts indicate the individual Hilbert spaces. Let us further simplify the notation before we proceed. We rewrite the above basis as:

$$\left[|\varphi_0^1\rangle \otimes |\varphi_0^2\rangle, |\varphi_0^1\rangle \otimes |\varphi_E^2\rangle, |\varphi_0^1\rangle \otimes |\varphi_P^2\rangle, |\varphi_E^1\rangle \otimes |\varphi_0^2\rangle, |\varphi_P^1\rangle \otimes |\varphi_0^2\rangle \right], \quad (6)$$

where the sub-scripts E and P refer to entangled and product states respectively. If there is a Schmidt rank two state in the null space it can be written as a linear combination of the above basis states. Using the fact that local projections cannot increase the Schmidt rank of a state, it readily follows that the coefficients of the basis states that are of the form $|\varphi_0^1\rangle \otimes |\varphi_P^2\rangle$ or $|\varphi_P^1\rangle \otimes |\varphi_0^2\rangle$ are zero. If any of these coefficients is not zero, then the reduced density matrix will have rank ≥ 3 . Therefore any Schmidt rank two state has to have the following form: $\alpha |\varphi_0^1\rangle \otimes |\varphi_E^2\rangle + \beta |\varphi_E^1\rangle \otimes |\varphi_0^2\rangle$. It is useful to analyze this explicitly. Let $|\psi\rangle$ be the Schmidt rank two state and hence it can be written as,

$$|\psi\rangle = \sum_{i=1}^{k-1} \alpha_i |\varphi_0^1\rangle \otimes |\varphi_{iE}^2\rangle + \sum_{i=1}^{k-1} \beta_i |\varphi_{iE}^1\rangle \otimes |\varphi_0^2\rangle + \gamma |\varphi_0^1\rangle \otimes |\varphi_0^2\rangle, \quad (7)$$

where the coefficients of the superposition are in general complex. On substituting the expressions for the states and rearranging it in the bipartite form one obtains

$$\sum_{j,l=0}^{k-1} \left(\sum_{i=1}^{k-1} \{ \alpha_i \lambda_{il} \lambda_{0j} + \beta_i \lambda_{0l} \lambda_{ij} \} + \gamma \lambda_{0j} \lambda_{0l} \right) |jl\rangle_A |jl\rangle_B. \quad (8)$$

The subscripts A, B are used to emphasize the Schmidt form of the above state. Note that (8) is already in a Schmidt decomposed form, where the terms in the parentheses correspond to the Schmidt coefficients. If the state is indeed of Schmidt rank two, then we must have all the coefficients but two equal to zero. This amounts to solving k^2 linear equations for $2k - 1$ variables. One can explicitly write down the above equations in a matrix form: $\mathbf{Ax} = \mathbf{y}$, where \mathbf{A} , \mathbf{x} , and \mathbf{y} are of dimensions $k^2 \times (2k - 1)$, $(2k - 1) \times 1$, and $k^2 \times 1$, respectively ($k \geq 3$). Moreover, \mathbf{y} has only two non-zero entries and $k^2 - 2$ zeros. The matrix \mathbf{A} can be shown to have the following property: Any submatrix of \mathbf{A} where *any* two of the rows are deleted (hence, the submatrix is of dimension $(k^2 - 2) \times (2k - 1)$) is still of full column rank. Hence, $\mathbf{x} = \mathbf{0}$, and it would imply that the above state has Schmidt rank zero. This completes the proof for two copies.

For n copies, the proof follows the same line as for two copies. The basis for the n -copy case of the null space is given by

$$\left\{ \binom{n}{m} |\varphi_0\rangle^{\otimes m} |\varphi_l\rangle^{\otimes n-m} \right\}, m = 0, \dots, n - 1 : l = 1, \dots, k - 1 \quad (9)$$

Following the same arguments as in the two copy case, one can obtain a similar set of linear equations. The number of equations is k^n and the number of variables can easily be counted and turns out to be $k^n - (k - 1)^n$; moreover, the right-hand-side of the equations (i.e., \mathbf{y}) has $k^n - 2$ zeros. Therefore, no matter how large n may be, number of equations is always greater than the number of variables, and one can show from the properties of the matrix that the set of linear equations does not have any non-trivial solution. \square

With the above result and the arguments provided in the beginning of the paper we can now directly state the following theorem.

Theorem 1 Let σ be a bipartite $d_1 \times d_2$ (where $d_1, d_2 \geq 3$) NPT state and let $|\varphi\rangle$ be a pure state of Schmidt rank equal to k ($3 \leq k \leq \min(d_1, d_2)$), such that $\langle \varphi | \sigma^{PT} | \varphi \rangle = -|\Lambda|$. Then for any $n \geq 1$, there exists an $\epsilon_n > 0$, such that the state

$$\rho(\epsilon) = \epsilon \sigma + \frac{(1 - \epsilon)}{d^2 - 1} (\mathbf{I} - |\varphi\rangle \langle \varphi|)^{PT} \quad (10)$$

is n -copy undistillable for $0 < \epsilon \leq \epsilon_n$.

We now show that the pairs of states (σ, φ) stipulated in Theorem 1 are fairly easy to construct. In our first method we will specify σ first, and then accordingly we will specify

$|\varphi\rangle$. In our second example, we will do just the opposite, i.e., we will fix an arbitrary state $|\varphi\rangle$ and based on the eigen decomposition of its partial transpose, we will construct σ .

Construction Method I: Choose any pure entangled state $|\psi\rangle$ of Schmidt rank m , $2 \leq m \leq d-1$ of the form $|\psi\rangle = \sum_{i=0}^{m-1} \beta_i |ii\rangle$, where $\sum_{i=0}^{m-1} |\beta_i|^2 = 1$ and the Schmidt coefficients β_i s are real and positive. Let $\sigma = |\psi\rangle\langle\psi|$. Since the Schmidt rank of $|\psi\rangle$ is at most $(d-1)$, therefore there is at least one product state that is orthogonal to the subspace spanned by the eigenvectors of σ^{PT} . Let $|\eta\rangle$ be such a product state. From the eigendecomposition of any pure state (see Eq.(2)) having Schmidt rank greater than or equal to two, we know that the eigenvectors corresponding to the negative eigenvalues are of Schmidt rank two. In particular, they are of the form $|\psi_{ij}^-\rangle = \frac{1}{\sqrt{2}}(|ij\rangle - |ji\rangle)$, $i < j$ with negative eigenvalue $\beta_i\beta_j$. Let $|\chi\rangle$ be one such eigenvector. Then, let $|\varphi\rangle = \sqrt{\alpha}|\psi_{ij}^-\rangle + \sqrt{1-\alpha}|\eta\rangle$. For instance, if $|\psi\rangle$ has Schmidt rank $(d-1)$, then $|\eta\rangle = |dd\rangle$. Clearly $|\varphi\rangle$ has Schmidt rank three in this case but the Schmidt rank can be greater than 3 if more than one mutually biorthogonal product states that are also orthogonal to σ^{PT} can be found. This would be determined by the Schmidt rank of $|\psi\rangle$. It is now obvious that $\langle\varphi|\sigma^{PT}|\varphi\rangle = -\alpha\beta_i\beta_j < 0$.

Construction Method II: Let us choose any arbitrary pure state $|\varphi\rangle$ that has Schmidt rank k , $3 \leq k \leq \min(d_1, d_2)$,

$$|\varphi\rangle = \sum_{i=0}^{k-1} \lambda_i |ii\rangle, \quad (11)$$

where $\sum_{i=0}^{k-1} |\lambda_i|^2 = 1$ and λ_i 's are real and positive. For any two operators A and B we have, $Tr(AB^{PT}) = Tr(A^{PT}B)$. For any σ , we therefore have, $\langle\varphi|\sigma^{PT}|\varphi\rangle = Tr(|\varphi\rangle\langle\varphi|\sigma^{PT}) = Tr(|\varphi\rangle^{PT}\langle\varphi|\sigma)$. It follows from Eq. (2) that if we choose σ as the convex combination of the eigenvectors with the negative eigenvalue of $|\varphi\rangle^{PT}\langle\varphi|$, then $\langle\varphi|\sigma^{PT}|\varphi\rangle$ will be negative. We therefore take the following representation of σ :

$$\sigma = \sum_{i,j=0, i<j}^{k-1} \alpha_{ij} |\psi_{ij}^-\rangle\langle\psi_{ij}^-|. \quad (12)$$

Then $\langle\varphi|\sigma^{PT}|\varphi\rangle = -\sum \alpha_{ij}\lambda_i\lambda_j < 0$.

Generalized Constructions

We can generalize our states in the following way. Let $m = \lfloor \frac{d}{k} \rfloor$, where $\lfloor x \rfloor$ is the "floor" operator denoting the largest integer less than or equal to x . Define the following states: $\rho_m(\epsilon) = \epsilon\sigma + \frac{(1-\epsilon)}{d^2-m} \left(\mathbf{I} - \sum_{i=1}^m |\varphi_i\rangle\langle\varphi_i| \right)^{PT}$, where $|\varphi_i\rangle$'s are pure entangled states of Schmidt rank $k \geq 3$ states such that each of them are in orthogonal subspaces. Note that it is not

necessary to have the Schmidt rank of the states to be equal but the choice was made for simplicity and convenience (notational). Clearly m is maximum for a given d when $k = 3$. The states are defined as follows $|\varphi_i\rangle = \sum_{j=k(i-1)}^{ki-1} \lambda_j^i |jj\rangle^i$. As before, σ may be chosen to be the convex combination of the states with negative eigenvalues in the eigen decomposition of the partial transpose of the pure states $|\varphi_i\rangle$. Note that $m = 1$ corresponds to the states in Theorem 1. One can then state the following generalization of Theorem 1.

Theorem 2. The states

$$\rho_m(\epsilon) = \epsilon\sigma + \frac{(1-\epsilon)}{d^2 - m} \left(\mathbf{I} - \sum_{i=1}^n |\varphi_i\rangle \langle \varphi_i| \right)^{PT} \quad (13)$$

for sufficiently small ϵ is n -copy undistillable for any $n \geq 1$.

Distance from the Maximally Mixed State

We next explore how these NPT n -copy undistillable states are distributed in the Hilbert space, and in particular, how far they are from the maximally mixed state. For any two quantum states ρ_1 and ρ_2 , the distance between the states is given by the Hilbert-Schmidt norm defined by $\|\rho_1 - \rho_2\| = \sqrt{\text{Tr}(\rho_1 - \rho_2)^2}$. Let us first note that the operator $\frac{1}{d^2 - m} \left(\mathbf{I} - \sum_{i=1}^n |\varphi_i\rangle \langle \varphi_i| \right)^{PT}$ is a PPT density matrix. The proof can be easily obtained by using Eq. (2). Since our n -copy undistillable states exist arbitrarily close to this state, it is sufficient to find the distance of this state from the maximally mixed one. Using the H-S norm, one can show that the distance is given by $\sqrt{\frac{m}{D(D-m)}}$. We also note that this distance is nothing but the distance of the maximally mixed state, $\frac{1}{D}\mathbf{I}$, from any normalized $(D-m)$ dimensional projector \mathbf{I}_{D-m} . Let us denote $r_m = \left\| \frac{1}{D}\mathbf{I} - \frac{\mathbf{I}_{D-m}}{D-m} \right\|$.

Theorem 3 For a bipartite quantum system in $d \otimes d$, the boundary of the balls of radius r_m for all $m = 1, \dots, \lfloor \frac{d}{k} \rfloor, k \geq 3$, around the maximally mixed state contains n -copy undistillable NPT states. For a given d , maximum number of such balls is obtained when $k = 3$.

It is instructive to analyze how close these states are relative to the largest separable ball, the radius of which has recently been obtained in Ref. [22], and is given by $\frac{1}{\sqrt{D(D-1)}}$. The result of Theorem 3 shows that the case $m = 1$ corresponds to the NPT n -copy undistillable states that lie on the boundary of the largest separable ball. This is as close as the states can be to the maximally mixed state. Let us now try to answer how far from the maximally mixed state these NPT finite copy undistillable states can be found. In our construction, for a given d , maximum r_m is obtained for $m = \lfloor \frac{d}{3} \rfloor$. This corresponds to a distance that grows as $\frac{1}{D^{1/4}}$.

Comparison with Previously Conjectured NPT BE states

We now point out a remarkable similarity of the class of states presented in this work with that obtained in Ref. [19]. It turns out that for certain choices of the parameters in their class of states and for a particular choice of $|\varphi\rangle$ in our case, the null space of the partial transposed operator is exactly the same! Let us denote the class of states in [19] as $\tilde{\rho}(c, \epsilon)$ (following their notation). When $\epsilon = 0$ and $c = 1/d(d+1)$,

$$\left(\tilde{\rho}\left(c = \frac{1}{d(d+1)}, \epsilon = 0\right)\right)^{PT} = \frac{1}{d^2 - 1} \left(\sum_{k=1}^{d-1} |\varphi_k\rangle \langle \varphi_k| + \sum_{k,l=0, k \neq l}^{d-1} |kl\rangle \langle kl| \right) \quad (14)$$

where, $|\varphi_k\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d-1} e^{\frac{2\pi i j k}{d}} |jj\rangle$, $k = 1.., d-1$. Going back to our class, let us choose $|\varphi\rangle$ to be the maximally entangled state of Schmidt rank d (i.e., $|\varphi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$), instead of our general original choice of any pure entangled state. If one expresses the identity operator as

$$\frac{1}{d^2} \mathbf{I} = \frac{1}{d^2} \left(|\varphi\rangle \langle \varphi| + \sum_{k=1}^{d-1} |\varphi_k\rangle \langle \varphi_k| + \sum_{k,l=0, k \neq l}^{d-1} |kl\rangle \langle kl| \right) \quad (15)$$

and substitutes Eq. (12) and $|\varphi\rangle$ in that of $\rho^{PT}(\epsilon = 0)$ (see Eq. (10), one obtains

$$\rho^{PT}(\epsilon = 0) = \frac{1}{d^2 - 1} \left(\sum_{k=1}^{d-1} |\varphi_k\rangle \langle \varphi_k| + \sum_{k,l=0, k \neq l}^{d-1} |kl\rangle \langle kl| \right).$$

The above similarity is striking considering the very different approaches adopted in the two construction methodologies.

Discussions and Concluding Remarks

We have shown that n -copy undistillable NPT states ($n \geq 1$) exist at infinitely many neighborhoods of the Hilbert space. Such states lie right on the surface of the largest separable ball (LSB); thus, they are as noisy as any inseparable state can be. They can also be found well outside of the LSB, where distillable and separable states coexist. Can the general approach adopted here lead to a proof of the existence of NPT BE states? Not in a straightforward manner: In our constructions, $\rho^{PT}(\epsilon = 0)$ has $D - 1$ identical nonzero eigenvalues, $\frac{1}{D-1}$. Hence, the function $f(\epsilon = 0, n) = \min_{SR(|\phi|)=2} \langle \phi | \left(\rho^{PT}(\epsilon = 0) \right)^{\otimes n} | \phi \rangle$ is bounded above by $(\frac{1}{D-1})^n$, and $\lim_{n \rightarrow \infty} f(\epsilon = 0, n) = 0$. Thus, we cannot claim that simple continuity arguments will yield the existence of states that are NPT but undistillable for any number of copies.

However, since $\rho(\epsilon = 0)$ is a separable state (and hence, undistillable), one should expect $\lim_{n \rightarrow \infty} f(\epsilon = 0, n) = 0$, and it provides no evidence that the provably n -copy undistillable states do not remain undistillable for any number of copies. In fact, *we conjecture that all the n -copy undistillable states constructed here are also truly NPT bound entangled states.* Moreover, we believe that even in our approach it is possible to show that there exists a neighborhood $0 \leq \epsilon \leq \epsilon_\infty$, where $\lim_{n \rightarrow \infty} f(\epsilon, n) = 0$; thus, proving that at least all the states in this neighborhood are also NPT BE states.

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