

Classification of nonasymptotic bipartite pure-state entanglement transformations

Somshubhro Bandyopadhyay,^{1,*} Vwani Roychowdhury,^{1,†} and Ujjwal Sen^{2,‡}

¹*Electrical Engineering Department, UCLA, Los Angeles, California 90095*

²*Department of Physics, Bose Institute, 93/1 Acharya Prafulla Chandra Road, Kolkata 700009, India*

(Received 25 July 2001; published 3 May 2002)

We show that deterministic and conclusive transformation properties of bipartite entanglement in the non-asymptotic scenario (when many but a finite number of copies of a source state are collectively manipulated) are fundamentally different from those in both the single-copy and asymptotic limits. For instance, by generalizing the notion of local comparability of entanglement in the single-copy case, we provide a complete classification of bipartite entanglement transformations in the nonasymptotic scenario. We also show that, unlike the asymptotic case, collective operations need not always be advantageous for the many-copy case. In particular, we show that (1) there exists a class of states for which the optimal conclusive transformation probability decreases exponentially with increasing number of copies, even if the source state has more entropy of entanglement, and (2) optimal conclusive transformation probability need not be a monotonic function of the number of copies.

DOI: 10.1103/PhysRevA.65.052315

PACS number(s): 03.67.-a, 03.65.Ud

Entanglement transformation addresses some fundamental concepts in quantum information theory, such as interconvertibility of different types of entanglement, and quantification of entanglement as a resource. Transformation properties are usually studied in two distinct regimes: (i) the asymptotic limit, where the parties collectively manipulate, in principle, an infinite amount of resources to attain the entropic bound [1,2], and (ii) the finite copy regime [3–7], where, as the name suggests, the parties manipulate only a finite number of shared entanglements. Although pure-state entanglement can be asymptotically diluted and concentrated with unit efficiency [1], this remarkable property does not hold in the finite-copy scenario. Moreover, recent results [5,6] have shown that transformation properties can be fundamentally different in the two regimes, and raised several key issues concerning the formulation of an appropriate entanglement measure.

Within the finite-copy regime, almost all known results and classifications apply to only the single-copy case, where both deterministic and conclusive transformations of bipartite states using local operations and classical communication (LOCC), with or without entanglement assistance (ELOCC), have been studied. The single-copy case is an extreme special case of the nonasymptotic regime, and it is unclear whether notions developed for this case are even justified or continue to hold in the many-copy setting. In this article, we investigate both deterministic and conclusive transformation properties of bipartite entanglement in the scenario when many but a finite number of copies of a source state are used to obtain as many exact copies of the target state under both LOCC and ELOCC. Our studies show that the many-copy case exhibits a number of unique transformation properties that are significantly different from those in both the single-copy and asymptotic limits.

For example, consider the fundamental notion of “incomparability” [4] of a given pair of entangled bipartite states, where a single copy of neither one of the given states can be converted to a copy of the other with probability one under LOCC. If a given state is deterministically LOCC transformable to another, then the entropy of its entanglement is at least as much as that of the target state, making the entanglements of the pair comparable; however, if a pair is incomparable, then it does not allow us to make any relative comparison of the entanglement of the two states. An interesting twist to the issue of entanglement comparability of pairs of states has been provided in [7], where it is shown that certain incomparable pairs are deterministically transformable (one way) in the presence of an auxiliary entanglement that remains intact in the process (catalysis). Thus, with auxiliary resources, the entanglement of otherwise single-copy incomparable pairs become comparable if they admit catalysts. This already suggests that the conventional notion of incomparability could be limited because of its restriction to the single-copy case, and that it needs to be generalized to capture the full power of the nonasymptotic scenario. Toward this end, one of the first questions we ask in this article is the following: Do single-copy incomparable pairs remain incomparable when collective operations are performed on multiple copies?

We answer the above question in the negative, and present the existence of states that are incomparable in the single-copy case but are nonetheless convertible in an exact and deterministic way if many copies are used in the transformation. In particular, we show that any given pair of states falls into either of the following two classes. (1) *k-copy LOCC comparable*, i.e., $|\psi\rangle^{\otimes k} \rightarrow |\phi\rangle^{\otimes k}$ or $|\phi\rangle^{\otimes k} \rightarrow |\psi\rangle^{\otimes k}$ by LOCC for some finite k with probability 1 but the states remain incomparable until $(k-1)$ copies, i.e., $|\psi\rangle^{\otimes n} \not\leftrightarrow |\phi\rangle^{\otimes n} \forall n \leq k-1$. (Note that there cannot exist any incomparable pair $\{|\psi\rangle, |\phi\rangle\}$ for which $|\psi\rangle^{\otimes k} \rightarrow |\phi\rangle^{\otimes k}$ and $|\phi\rangle^{\otimes k} \rightarrow |\psi\rangle^{\otimes k}$ hold simultaneously. This follows from the fact that if both transformations hold, then the Schmidt coefficients of the states $|\psi\rangle$ and $|\phi\rangle$ must be equal, which is a contradiction.) (2) *Strongly incomparable*, that is, $\{|\psi\rangle^{\otimes k}, |\phi\rangle^{\otimes k}\}$ remain in-

*Electronic address: som@ee.ucla.edu

†Electronic address: vwani@ee.ucla.edu

‡Electronic address: dhom@bosemain.boseinst.ernet.in

comparable for any finite k even under ELOCC. We go on to provide a necessary condition for a pair to be k -copy LOCC comparable for a finite k , and a sufficient condition for a pair to be strongly incomparable in $d \times d$ for all $d \geq 3$, which provides an easy method of generating such states in $d \times d$.

The above results might indicate that using many copies can be always beneficial, but we show that this is not the case. Moving from deterministic cases to conclusive ones, we find two intriguing features. (1) An increase in the number of copies can result in an exponential decrease in the conclusive transformation probability, i.e., $p_{\max}(|\psi\rangle^{\otimes k} \rightarrow |\phi\rangle^{\otimes k}) \sim c^k$, $c < 1$, under both LOCC and ELOCC, even though the source state $|\psi\rangle$ has *more* entropy of entanglement. (2) The optimal probability of a conclusive conversion $p_{\max}(|\psi\rangle^{\otimes k} \rightarrow |\phi\rangle^{\otimes k})$ may not be a monotonic function of the number of copies k . These results show the surprising fact that collective manipulations need not necessarily be advantageous in the case of exact and multicopy conclusive transformations.

A bipartite $d \times d$ pure quantum state $|\psi\rangle$ is usually represented as $|\psi\rangle = \sum_{i=1}^d \sqrt{\alpha_i} |i\rangle |i\rangle$ with ordered Schmidt coefficients, $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_d \geq 0$, which are also the eigenvalues of the reduced density operator. Since the eigenvalues determine the existence or nonexistence of transformations to be studied in this article, it is convenient for us to denote the state itself by its eigenvalue vector: $\psi = (\alpha_1, \dots, \alpha_d)$. The central tool in proving our results is due to Nielsen [4]: A bipartite pure state $|\psi\rangle$ transforms to another state $|\phi\rangle$ using LOCC with probability 1 if and only if ψ is majorized by ϕ (written $\psi \prec \phi$), that is, if and only if for each m in the range $1, \dots, d$,

$$\sum_{i=1}^m \psi^{(i)} \leq \sum_{i=1}^m \phi^{(i)}. \quad (1)$$

Incomparable pairs are those for which the majorization condition is violated by the concerned states [4]. For instance, one can easily check that the states $\psi = (0.4, 0.36, 0.14, 0.1)$ and $\phi = (0.5, 0.25, 0.25, 0)$ are incomparable. However, as noted earlier, the single-copy scenario is a special case of the nonasymptotic regime, where we allow many copies to take part in the transformation. We now show that it is unnecessary to conclude that the entanglement of two states is “incomparable” based only on their properties in the single-copy case, and that the notion of incomparability can be naturally generalized to the case involving many copies. In particular, we show the existence of states $\{|\psi\rangle, |\phi\rangle\}$ that have the following properties: (1) $|\psi\rangle \leftrightarrow |\phi\rangle$ under LOCC; and (2) $|\psi\rangle^{\otimes k} \rightarrow |\phi\rangle^{\otimes k}$ with probability 1 by LOCC for some $k > 1$.

Consider the single-copy incomparable states $\psi = (0.4, 0.36, 0.14, 0.1)$ and $\phi = (0.5, 0.25, 0.25, 0)$. The eigenvalue vectors of the tensor product states for two copies of the states are

$$\begin{aligned} \psi^{\otimes 2} = & (0.16, 0.144, 0.144, 0.1296, 0.056, 0.056, \\ & 0.0504, 0.0504, 0.04, 0.04, 0.036, 0.036, 0.0196, 0.014, \\ & 0.014, 0.001), \end{aligned} \quad (2)$$

$$\begin{aligned} \phi^{\otimes 2} = & (0.25, 0.125, 0.125, 0.125, 0.125, 0.0625, \\ & 0.0625, 0.0625, 0.0625, 0, 0, 0, 0, 0). \end{aligned} \quad (3)$$

It is easy to check that $\psi^{\otimes 2} \prec \phi^{\otimes 2}$, implying that the transformation $|\psi\rangle^{\otimes 2} \rightarrow |\phi\rangle^{\otimes 2}$ can in fact be realized by LOCC with certainty 0. Hence, $\psi = (0.4, 0.36, 0.14, 0.1)$ and $\phi = (0.5, 0.25, 0.25, 0)$ are two-copy LOCC comparable. Examples of k -copy comparable states with different values of k include the following.

(a) *Three-copy LOCC comparable:* $\{\psi = (0.4, 0.4, 0.1, 0.1); \phi = (0.5, 0.27, 0.23, 0)\}$. Observe that $p_{\max}(|\psi\rangle \rightarrow |\phi\rangle) \cong 87\%$ and $p_{\max}(|\psi\rangle^{\otimes 2} \rightarrow |\phi\rangle^{\otimes 2}) \cong 99\%$. Therefore, as one might expect, the transformation probability increases with number of copies.

(b) *Six-copy LOCC comparable pair:* $\{\psi = (0.4, 0.4, 0.1, 0.1); \phi = (0.48, 0.27, 0.25, 0)\}$.

Existence of such exact and deterministic k -copy transformations might prove to be of some practical value as well. Take, for instance, the two-copy LOCC comparable pair, discussed above. This pair is catalyzable, and one can verify that the 2×2 state $\chi = (0.6, 0.4)$ is a valid catalyst for the pair. Thus to obtain two copies of $|\phi\rangle$ from two copies of $|\psi\rangle$, one needs two such entanglement assisted transformations. But we have already shown that the same goal can be reached by a single collective transformation without any catalyst.

We would now like to ask what conditions need to be satisfied for k -copy LOCC comparability, i.e., whether a transformation $|\psi\rangle^{\otimes k} \rightarrow |\phi\rangle^{\otimes k}$ (or vice versa) is always possible for some k by LOCC (or even by ELOCC)? We give a necessary condition for such transformations to exist.

Lemma 1. Let $|\psi\rangle$ and $|\phi\rangle$ be $d \times d$ states, with ordered Schmidt coefficients $\{\alpha_j\}, \{\beta_j\}$, $1 \leq j \leq d$, respectively. Then there exists some $k > 1$ such that $|\psi\rangle^{\otimes k} \rightarrow |\phi\rangle^{\otimes k}$ under LOCC only if $\alpha_1 \leq \beta_1$ and $\alpha_d \geq \beta_d$. The same necessary condition also holds for $|\psi\rangle^{\otimes k} \rightarrow |\phi\rangle^{\otimes k}$ under ELOCC.

Proof. If there is some k such that $|\psi\rangle^{\otimes k} \rightarrow |\phi\rangle^{\otimes k}$, then from Nielsen’s theorem it follows that $\alpha_1^k \leq \beta_1^k$ and $1 - \alpha_d^k \leq 1 - \beta_d^k$ which implies $\alpha_1 \leq \beta_1$ and $\alpha_d \geq \beta_d$. This proves the first part of the lemma.

It has been shown in Ref. [7] that $|\psi\rangle \rightarrow |\phi\rangle$ under ELOCC only if $\alpha_1 \leq \beta_1$ and $\alpha_d \geq \beta_d$. It is straightforward to show that a similar condition holds for $|\psi\rangle^{\otimes k} \rightarrow |\phi\rangle^{\otimes k}$ under ELOCC. ■

From Lemma 1 it follows that 3×3 incomparable states remain incomparable even if multiple copies are available. This is due to the fact that for 3×3 incomparable states, if $\alpha_1 < \beta_1$, then $\alpha_3 < \beta_3$. Hence incomparable states in 3×3 are neither catalyzable even with multiple copies nor multiple-copy transformable.

In general, an incomparable pair $\{|\psi\rangle, |\phi\rangle\}$ is said to be strongly incomparable if the pair is noncatalyzable even with multiple copies, i.e., $|\psi\rangle^{\otimes k} \not\leftrightarrow |\phi\rangle^{\otimes k}$ under ELOCC for all k . Strongly incomparable pairs are obviously k -copy LOCC incomparable for all k . The following result provides a sufficient condition for strong incomparability and gives an easy

method for constructing strongly incomparable states in $d \times d$ for any $d \geq 3$.

Theorem 1. Let $|\psi\rangle$ and $|\phi\rangle$ be $d \times d$ states, with ordered Schmidt coefficients $\{\alpha_j\}, \{\beta_j\}$, $1 \leq j \leq d$, respectively. A sufficient condition that they form a strongly incomparable pair is $\alpha_1 < \beta_1$ and $\alpha_d < \beta_d$ or $\alpha_1 > \beta_1$ and $\alpha_d > \beta_d$.

Proof. From Lemma 1 it follows that if $\alpha_1 < \beta_1$ and $\alpha_d < \beta_d$ or $\alpha_1 > \beta_1$ and $\alpha_d > \beta_d$ then $|\psi\rangle^{\otimes k} \leftrightarrow |\phi\rangle^{\otimes k}$ under ELOCC for all k . Hence the proof. ■

Next we turn our attention to the case of conclusive LOCC and ELOCC transformations. If $\{|\psi\rangle, |\phi\rangle\}$ is strongly incomparable, then there does not exist any local strategy such that k copies of $|\psi\rangle$ can be converted into k copies of $|\phi\rangle$ or vice versa with certainty under LOCC and even by ELOCC. Hence, for any k , the transformation $|\psi\rangle^{\otimes k} \rightarrow |\phi\rangle^{\otimes k}$ (or the reverse one) is necessarily conclusive. We would now like to know how the transformation probability changes with the number of copies when the conversion is conclusive.

For conclusive transformations, if the source state has at least as many nonzero Schmidt coefficients as the target state, then a conclusive transformation is possible with the optimal probability given by $p_{\max}(|\psi\rangle \rightarrow |\phi\rangle) = \min_{1 \leq l \leq d} [E_l(|\psi\rangle)/E_l(|\phi\rangle)]$, where $E_l(|\psi\rangle) = 1 - \sum_{i=1}^{l-1} \alpha_i$ [5].

Lemma 2. Let $|\psi\rangle$ and $|\phi\rangle$ be $d \times d$ states with ordered Schmidt coefficients $\{\alpha_j\}, \{\beta_j\}$, $1 \leq j \leq d$, respectively, and $|\text{MES}\rangle$ be a maximally entangled state in $d \times d$. Then $p_{\max}(|\psi\rangle \rightarrow |\text{MES}\rangle) < p_{\max}(|\phi\rangle \rightarrow |\text{MES}\rangle)$ if and only if $\alpha_d < \beta_d$.

A proof follows by noting that $p_{\max}(|\psi\rangle \rightarrow |\text{MES}\rangle) = d\alpha_d$ and $p_{\max}(|\phi\rangle \rightarrow |\text{MES}\rangle) = d\beta_d$ [3,5]. We now show that the condition $p_{\max}(|\psi\rangle \rightarrow |\text{MES}\rangle) < p_{\max}(|\phi\rangle \rightarrow |\text{MES}\rangle)$ is sufficient to ensure that the optimal probability of a conclusive transformation can never increase with the number of copies.

Theorem 2. If $p_{\max}(|\psi\rangle \rightarrow |\text{MES}\rangle) < p_{\max}(|\phi\rangle \rightarrow |\text{MES}\rangle)$, then $p_{\max}(|\psi\rangle^{\otimes k} \rightarrow |\phi\rangle^{\otimes k})$ falls off exponentially (under both LOCC and ELOCC) with k , the number of copies.

Proof. First consider the LOCC case. It follows from Lemma 2 that $p_{\max}(|\psi\rangle \rightarrow |\phi\rangle) \leq \alpha_d/\beta_d < 1$. Hence $p_{\max}(|\psi\rangle^{\otimes k} \rightarrow |\phi\rangle^{\otimes k}) = \min_{1 \leq d^k} [E_{d^k}(|\psi\rangle^{\otimes k})/E_{d^k}(|\phi\rangle^{\otimes k})] \leq E_{d^k}(|\psi\rangle^{\otimes k})/E_{d^k}(|\phi\rangle^{\otimes k}) = (\alpha_d/\beta_d)^k < 1$, $\forall k \geq 2$. The proof for the ELOCC case from the observation that if $p_{\max}(|\psi\rangle \rightarrow |\text{MES}\rangle) < p_{\max}(|\phi\rangle \rightarrow |\text{MES}\rangle)$, then $p_{\max}(|\psi\rangle \otimes |\chi\rangle \rightarrow |\text{MES}\rangle) < p_{\max}(|\phi\rangle \otimes |\chi\rangle \rightarrow |\text{MES}\rangle)$ for any auxiliary state $|\chi\rangle$; hence, one can use the same proof as for the LOCC case. ■

Two comments on the implications of Lemma 2 and Theorem 2 are appropriate here.

(i) From Lemma 2 it follows that if $p_{\max}(|\psi\rangle \rightarrow |\phi\rangle) \leq \alpha_d/\beta_d < 1$ then $p_{\max}(|\psi\rangle \rightarrow |\text{MES}\rangle) < p_{\max}(|\phi\rangle \rightarrow |\text{MES}\rangle)$. Hence, it follows from Theorem 2 that if $p_{\max}(|\psi\rangle \rightarrow |\phi\rangle) \leq \alpha_d/\beta_d < 1$, then $p_{\max}(|\psi\rangle^{\otimes k} \rightarrow |\phi\rangle^{\otimes k})$ falls off exponentially with the number of copies.

(ii) From Theorem 2 it follows that if $\alpha_1 < \beta_1$ and $\alpha_d < \beta_d$ then the states are strongly incomparable. Let us also note that in Ref. [7] it was shown that if $p_{\max}(|\psi\rangle \rightarrow |\phi\rangle) = \alpha_d/\beta_d$, then the probability of conclusive transformation

cannot be increased in the presence of any catalyst. Thus, our results show that there exist incomparable pairs for which the conclusive transformation probability cannot be improved in the presence of any catalyst and using multiple copies the probability falls off exponentially.

Note that the condition $p_{\max}(|\psi\rangle \rightarrow |\text{MES}\rangle) < p_{\max}(|\phi\rangle \rightarrow |\text{MES}\rangle)$ might be satisfied even though $E(|\psi\rangle) > E(|\phi\rangle)$ where E is the entropy of entanglement. Consider the following incomparable pair in 3×3 , which we know to be strongly incomparable: $\{\zeta = (0.4, 0.4, 0.2); \omega = (0.5, 0.25, 0.25)\}$. We are interested in how the conclusive transformation probability $p_{\max}(|\text{source}\rangle^{\otimes k} \rightarrow |\text{target}\rangle^{\otimes k})$ scales with k , k being arbitrarily large but finite. Let us first collect the following facts about the above pair.

(1) $E(\zeta) > E(\omega)$, which means that in the asymptotic limit $|\zeta\rangle$ generates a larger number of maximally entangled states as compared to $|\omega\rangle$.

(2) Let $|\text{MES}\rangle$ be a maximally entangled state in 3×3 . Then $p_{\max}(|\zeta\rangle \rightarrow |\text{MES}\rangle) < p_{\max}(|\omega\rangle \rightarrow |\text{MES}\rangle)$, which means that given a large but finite number of copies we can obtain more maximally entangled states from $|\omega\rangle$ when we use a conclusive conversion protocol.

Let us now consider the scenario when many copies are used to transform the states among themselves.

Case 1. $|\zeta\rangle, |\omega\rangle$ are the source and target states, respectively. First note that $p_{\max}(|\zeta\rangle \rightarrow |\omega\rangle) = \alpha_3/\beta_3 = 4/5$. Hence, $p_{\max}(|\zeta\rangle^{\otimes k} \rightarrow |\omega\rangle^{\otimes k}) \leq (\alpha_3/\beta_3)^k = (4/5)^k$. Therefore, for large k , $p_{\max}(|\zeta\rangle^{\otimes k} \rightarrow |\omega\rangle^{\otimes k})$ falls off exponentially to zero even though $E(|\zeta\rangle) > E(|\omega\rangle)$. Since the conversion is conclusive, a successful conversion always results in an exact outcome. At this point, it is instructive to analyze this result by comparing it to an asymptotic conversion. Note that there is no contradiction with the result of Bennett *et al.* [1]. To see this, consider what happens in an asymptotic conversion. It was shown in Ref. [1] that in an asymptotic conversion, the yield approaches $E(\zeta)/E(\omega)$, the fidelity approaching 1 and the success of probability also approaching 1 in the limit of large k . Since $E(\zeta)/E(\omega) > 1$, in the limit $k \rightarrow \infty$, we would obtain at least as many copies of $|\omega\rangle$ with fidelity approaching unity. This apparent contradiction is resolved at once by noting that for any finite k , however large, the conversion is always approximate and the success probability is always less than 1.

Case 2. $|\omega\rangle, |\zeta\rangle$ are the source and target states, respectively. We present this case through numerical results that indicate a rather surprising feature. We find that, as we keep increasing the number of copies, the transformation probability shows an approximately damped oscillatory behavior (see Fig. 1). This clearly shows that the transformation probability may not be a monotonic function of the number of copies. Note that the maximum transformation probability occurs when $k=3$. So the transformation probability increases to maximum at $k=3$ and then decays in an oscillatory fashion. What is curious in this behavior is the lack of monotonicity.

More surprisingly, such nonmonotonic behavior is observed even in the case of k -copy LOCC comparable states, when we examine the pairs $\{|\psi\rangle^m, |\phi\rangle^m\}$, where $m > k$. Clearly, if $m = kl$, $l \geq 2$, then the pairs are again comparable, and the corresponding LOCC transformations occur with

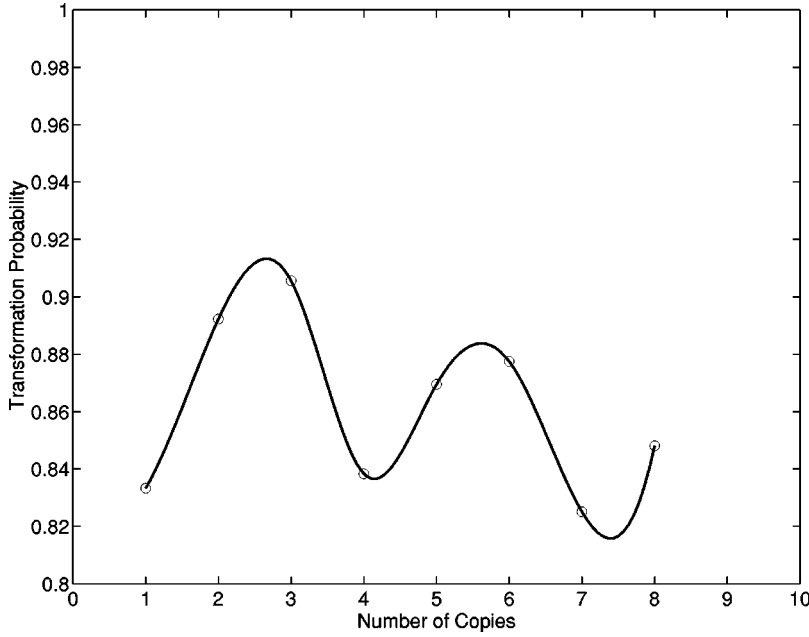


FIG. 1. In the nonasymptotic regime conclusive transformation probability oscillates with the number of copies of the source state.

probability 1. However, in response to a conjecture stated in an earlier draft of this paper on the LANL web site [8], Ref. [9] provided examples of k -copy comparable pairs, where, for example, the pair $\{|\psi\rangle^{\otimes(k+1)}, |\phi\rangle^{\otimes(k+1)}\}$ is no longer comparable (i.e., the probability of transformation is < 1). Hence, as a function of the number of copies m , the transformation probability reaches unity at regular intervals (i.e., at multiples of k), but in between these points the optimal probability exhibits a complex nonmonotonic behavior.

To summarize, we have investigated both deterministic and conclusive transformation properties of bipartite entanglement in the scenario when many but a finite number of copies of a source state are collectively manipulated. We have shown that such a nonasymptotic many-copy case exhibits several transformation properties that are significantly different from those in both the single-copy and asymptotic limits. For example, we showed that the notion of incomparability is restricted if only the single-copy scenario is considered, and we introduced generalized notions of comparable and incomparable pairs of states that are appropriate for deterministic transformations (with or without auxiliary entanglement assistance) in the nonasymptotic case. We also demonstrated, unlike the asymptotic case, collective operations on an increasing number of copies of the source states need not always be advantageous for the many-copy case. In

particular, we showed that (1) there exists a class of states for which the optimal conclusive transformation probability decreases exponentially with increasing number of copies, even if the source state has more entropy of entanglement, and (2) optimal conclusive transformation probability need not be a monotonic function of the number of copies.

The study of transformation properties in the nonasymptotic case opens up a number of avenues of research. For example, should one also define the concept of k -copy ELOCC comparable states? In other words, are there pairs of states that are k -copy LOCC comparable, but the pairs $\{|\phi\rangle^{\otimes k'}, |\psi\rangle^{\otimes k'}\}$ become ELOCC comparable for some $k' < k$? Are there stronger necessary conditions for states to be k -copy LOCC comparable than those derived in Lemma 1? Similarly, what are the necessary and sufficient conditions for states to be strongly incomparable?

We would like to thank Tal Mor and Guruprasad Kar for useful discussions. The work of S.B. and V.R. was sponsored in part by the Defense Advanced Research Projects Agency (DARPA) Project No. MDA 972-99-1-0017 and in part by the U.S. Army Research Office/DARPA under Contract/Grant No. DAAD 19-00-1-0172. This work is also supported in part by the NSF under Grant No. EIA-011349. U.S. acknowledges partial support by the Council of Scientific and Industrial Research, Government of India, New Delhi.

-
- [1] C.H. Bennett, H.J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
 - [2] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, Phys. Rev. A **54**, 3824 (1996).
 - [3] H.-K. Lo and S. Popescu, Phys. Rev. A **63**, 022301 (2001).
 - [4] M.A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999).
 - [5] G. Vidal, Phys. Rev. Lett. **83**, 1046 (1999).

- [6] G. Vidal, J. Mod. Opt. **47**, 355 (2000).
- [7] D. Jonathan and M.B. Plenio, Phys. Rev. Lett. **83**, 3566 (1999).
- [8] S. Bandyopadhyay, V. Roychowdhury, and U. Sen, e-print quant-ph/0103131.
- [9] D. Leung and J. Smolin, e-print quant-ph/0103158.