Efficient entanglement-assisted transformation for bipartite pure states

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We show that entanglement-assisted transformations of bipartite entangled states can be more efficient than catalysis [Jonathan and Plenio, Phys. Rev. Lett. **83**, 3566 (1999)], i.e., given two incomparable bipartite states not only can the transformation be enabled by performing collective operations with an auxiliary entangled state, but the entanglement of the auxiliary state itself can be enhanced. We refer to this phenomenon as *supercatalysis*. We provide results on the properties of supercatalysis and its relationship with catalysis. In particular, we obtain a useful necessary and sufficient condition for catalysis, and provide several sufficient conditions for supercatalysis and study the extent to which entanglement of the auxiliary state can be enhanced via supercatalysis.

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One of the primary goals of quantum information theory [1] is efficient manipulation of quantum entanglement shared among spatially separated parties, each of whom possesses only a subsystem of the entire entangled state [2]. Such distributed entanglement, as a resource, is a critical component of quantum information protocols, such as quantum teleportation [3], superdense coding [4], and distributed computing algorithms [5]. Since the underlying entangled state is spatially distributed, any entanglement manipulation is necessarily constrained to be carried out with local operations and classical communication among the parties (LOCC). The properties and classifications of both deterministic and probabilistic/conclusive LOCC transformations have been pursued vigorously in the recent past [6–12].

A surprising feature that sets apart entanglement from usual physical resources is its capacity to enable, without being consumed, transformations that are impossible under *deterministic* LOCC [9]. This property is very similar to that of catalysts in chemical reactions and is aptly termed as *entanglement catalysis*. It has also been shown that the probability of a conclusive conversion can be enhanced in the presence of a catalyst, when a deterministic conversion is not possible [9]. Another instance where entanglement is useful in a sense similar to catalysis (i.e., not being consumed), is *partial recovery* of entanglement. In this case, the entanglement lost in an LOCC manipulation is partially recovered using an auxiliary entanglement and performing collective operations [10,11].

We show that the above two features of entanglement can be exploited simultaneously and that entanglement assisted LOCC transformations can be more efficient than catalysis [9]. In particular, given two incomparable states (i.e., states that are not LOCC transformable with certainty), not only can the transformation be enabled by performing collective operations with an auxiliary entangled state, but the entanglement of the auxiliary state itself can be enhanced. Such simultaneous enabling of deterministic LOCC impossible transformations, and reduction of the overall loss in entanglement is not possible under catalysis [9]. We refer to this phenomenon as *supercatalysis*. In this paper, we study the properties of supercatalysis and its relationship with catalysis, obtain a useful and succinct necessary and sufficient condition for catalysis, and sufficient conditions for supercatalysis.

All the transformations that we consider in this paper are deterministic, i.e., occur with probability 1, and are in the finite copy regime. We represent an $n\times n$ bipartite pure entangled state $|\psi\rangle$, as

$$|\psi\rangle=\sum_{i=1}^{n}\sqrt{\alpha_i}|i\rangle|i\rangle,$$

where $\alpha_1\geqslant\alpha_2\geqslant\cdots\geqslant\alpha_n$, are the Schmidt coefficients or eigenvalues of the reduced density matrices. Also, let λ_ψ denote the vector of the ordered eigenvalues. Then, it follows from Nielsen's result [7] that for any two given $n\times n$ states $|\psi\rangle=\Sigma_{i=1}^{n}\sqrt{\alpha_i}|i\rangle|i\rangle$ and $|\varphi\rangle=\Sigma_{j=1}^{n}\sqrt{\beta_j}|j\rangle|j\rangle$, $|\psi\rangle\rightarrow|\varphi\rangle$ with probability 1 under LOCC, if and only if, λ_ψ is *majorized* by λ_φ, (denoted as $\lambda_\psi\prec\lambda_\varphi$), i.e.,

$$\sum_{i=1}^{m}\alpha_i\leqslant\sum_{i=1}^{m}\beta_i,\quad\text{for every}\quad m=1,\ldots,n-1. \tag{1}$$

Note that the above inequality is satisfied trivially when $m=n$, since both sides equal 1. In the rest of the paper, for the sake of convenience, instead of representing a bipartite state $|\psi\rangle=\Sigma_{i=1}^{n}\sqrt{\alpha_i}|i\rangle|i\rangle$, we shall represent it simply by the vector of its eigenvalues $|\psi\rangle=(\alpha_1,\alpha_2,\ldots,\alpha_n)$.

Consider the following pair of 4×4 *bipartite incomparable* states,

$$|\psi\rangle=(0.4,0.36,0.14,0.1), \tag{2}$$

$$|\phi\rangle=(0.5,0.25,0.25,0.0), \tag{3}$$

for which an auxiliary entangled state $|\chi\rangle=(0.65,0.35)$ is a catalyst, i.e., the transformation $|\psi\rangle\otimes|\chi\rangle\rightarrow|\phi\rangle\otimes|\chi\rangle$ can be realized deterministically under LOCC. An example of supercatalysis lies in showing the existence of a state, say $|\omega\rangle=\Sigma_{i=1}^{k}\sqrt{\gamma_i}|i\rangle|i\rangle$, such that $|\psi\rangle\otimes|\chi\rangle\rightarrow|\phi\rangle\otimes|\omega\rangle$ by

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LOCC with probability 1, where $E(\omega) > E(\chi)$, E being the entropy of entanglement [e.g., $E(|\omega\rangle) = -\Sigma_{i=1}^{k}\gamma_i \ln(\gamma_i)$]. Let $|\omega\rangle = (0.55,0.45)$. Note that $E(\omega) > E(\chi)$. The corresponding eigenvalue vectors $\lambda_{\psi\otimes\chi}, \lambda_{\phi\otimes\omega}$ are given by

$$(0.26,0.234,0.14,0.126,0.091,0.065,0.049,0.035), \tag{4}$$

$$(0.275,0.225,0.1375,0.1375,0.1125,0.1125,0,0), \tag{5}$$

respectively. It can be easily verified that $\lambda_{\psi\otimes\chi} < \lambda_{\phi\otimes\omega}$, and hence, the transformation $|\psi\rangle\otimes|\chi\rangle \to |\phi\rangle\otimes|\omega\rangle$ is possible under LOCC with certainty. As the final state $|\omega\rangle$, of the initial auxiliary state $(|\psi\rangle)$, is more entangled than its initial one, supercatalysis is clearly more efficient than catalysis. An equivalent interpretation of the underlying phenomenon is that supercatalysis, in addition to enabling the transformation, reduces the overall loss in entanglement. In catalysis the net entanglement lost is just the difference between entanglement of the parent states. Supercatalysis reduces this loss by an amount $\delta = E(\omega) - E(\chi)$. One can think of several innovative uses of supercatalysis, and a particular scenario, where resources are limited and constrained is outlined next. For instance, consider a scenario where we are given two copies of the source state, say $|\psi\rangle = (0.4,0.4,0.1,0.1)$ and we wish to obtain the target states $|\phi_1\rangle = (0.5,0.25,0.25,0)$ and $|\phi_2\rangle = (0.48,0.27,0.25,0)$, respectively. One can easily verify that all the following pairs are incomparable: $\{|\psi\rangle,|\phi_1\rangle\}$, $\{|\psi\rangle,|\phi_2\rangle\}$, and $\{|\psi\rangle\otimes|\psi\rangle,|\phi_1\rangle\otimes|\phi_2\rangle\}$. Since both direct individual LOCC transformations, and the collective LOCC transformation are ruled out, we require either two different catalyst states, one for each pair, or a single catalyst that can work for both the transformations. Suppose the entanglement supplier fails to provide two catalysts for the two pairs or a common catalyst that may work for both of them, but instead provides only one, say $|\chi\rangle = (0.625,0.375)$, which is useful only to carry out a single transformation, i.e.,

$$|\psi\rangle\otimes|\chi\rangle \to |\phi_1\rangle\otimes|\chi\rangle, \tag{6}$$

$$|\psi\rangle\otimes|\chi\rangle \nrightarrow |\phi_2\rangle\otimes|\chi\rangle. \tag{7}$$

It will be clear from the following discussions as to why the given catalyst state does not work for the second transformation. It is not entangled enough. In situations such as this supercatalysis can provide a solution.

Step 1. (Supercatalysis) Perform a supercatalytic transformation involving the incomparable pair $\{|\psi\rangle,|\phi_1\rangle\}$ and the given auxiliary state $|\chi\rangle$,

$$|\psi\rangle\otimes|\chi\rangle \to |\phi_1\rangle\otimes|\omega\rangle, \tag{8}$$

where the new state is $|\omega\rangle = (\frac{8}{13},\frac{5}{13})$, with $E(\omega) > E(\chi)$.

Step 2. (Catalysis) The new improved auxiliary state $|\omega\rangle$, is now sufficiently entangled to act as a legitimate catalyst for the second incomparable pair, and one can easily check that the transformation

$$|\psi\rangle\otimes|\omega\rangle \to |\phi_2\rangle\otimes|\omega\rangle, \tag{9}$$

can indeed be realized under LOCC with probability 1.

The above example shows that one might be able to perform a series of transformations with limited ancillary resources by improving the catalyst appropriately at every step to make it useful for subsequent transformations.

In the rest of this paper, we provide results on the existence of supercatalysts for given pairs of incomparable states, and study its relationship with catalyst states. For example, given a supercatalytic transformation what can we say about the "catalytic" properties of the auxiliary states? Clearly, if the two auxiliary states (i.e., the initial and the final auxiliary states) involved in the supercatalysis transformation are in 2×2, then they are *both* catalysts as well. However, whether such a property is always true for higher-dimensional auxiliary states is left as an open problem, and the following result provides a sufficient condition.

Proposition 1. Let $|\chi\rangle$ and $|\phi\rangle$ be the initial and final entangled states facilitating supercatalysis of the incomparable pair $\{|\psi\rangle,|\phi\rangle\}$. If $|\omega\rangle \to |\chi\rangle$ under LOCC, then $\{|\chi\rangle,|\omega\rangle\}$ are also catalysts for the incomparable pair $\{|\psi\rangle,|\phi\rangle\}$.

Proof. If $|\omega\rangle \to |\chi\rangle$, then we have the following transformations: (1) $|\psi\rangle\otimes|\chi\rangle \to |\phi\rangle\otimes|\omega\rangle \to |\phi\rangle\otimes|\chi\rangle$ and (2) $|\psi\rangle\otimes|\omega\rangle \to |\psi\rangle\otimes|\chi\rangle \to |\phi\rangle\otimes|\omega\rangle$ from which it follows that $\{|\chi\rangle,|\omega\rangle\}$ are catalysts for the incomparable pair $\{|\psi\rangle,|\phi\rangle\}$. ■

As an immediate implication of the above proposition, we show the following bound on the entanglement of the final auxiliary state $|\omega\rangle$.

Corollary 1. For a given incomparable pair $\{|\psi\rangle,|\phi\rangle\}$ in $n\times n$, let $k\times k$ states $\{|\chi\rangle,|\omega\rangle\}$ be the corresponding supercatalysts [i.e., $|\psi\rangle\otimes|\chi\rangle \to |\phi\rangle\otimes|\omega\rangle$ with probability 1 under LOCC, and $E(|\omega\rangle) > E(|\chi\rangle)$]. The improved state $|\omega\rangle$ can never be a maximally entangled state in $k\times k$.

Proof. Let $|\omega\rangle$ be a maximally entangled state in $k\times k$. Then $|\omega\rangle \to |\chi\rangle$. Therefore, by lemma 1, $|\omega\rangle$ and $|\chi\rangle$ are the catalysts for the given incomparable pair. But a maximally entangled state cannot be a catalyst [9]. Hence the proof. ■

We next investigate the presence of supercatalysis when there exist catalytic states for a given pair of incomparable parent states. The associated *formalism turns out to be extremely useful*. It provides a general framework and a necessary and sufficient condition for constructing catalytic states, leads to sufficient conditions for supercatalysis and allows us to determine meaningful bounds on the enhanced entanglement of the auxiliart state. Given an incomparable pair $\{|\psi\rangle,|\phi\rangle\}$, with eigenvalue vectors $\lambda_\psi = \{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ and $\lambda_\phi = \{\beta_1,\beta_2,\ldots,\beta_n\}$, let $|\chi(\mathcal{P})\rangle$ be a $k\times k$ catalyst with the eigenvalue vector

$$\lambda_\chi = \mathcal{P} = \left\{ p_1,p_2,\ldots,p_k = 1 - \sum_{i=1}^{k-1} p_i \right\},$$

where $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_{k-1} \geqslant p_k$. The proof of the following lemma provides a constructive computational procedure for determining all possible such $k\times k$ catalytic states.

Lemma 1. The set of all $k\times k$ catalytic states for any given $n\times n$ pair of incomparable states $\{|\psi\rangle,|\phi\rangle\}$, is either empty, or a union of a finite number of polyhedra in dimension $\leqslant (k-1)$.

Proof. Since we want auxiliary states,

$$\left|\chi\left(p_1,p_2,\ldots,p_k=1-\sum_{i=1}^{k-1}p_i\right)\right\rangle,$$

such that $\lambda_{\psi\otimes\chi(\mathcal{P})}<\lambda_{\phi\otimes\chi(\mathcal{P})}$, the set of all possible $\mathcal{P}=(p_1,p_2,\ldots,p_{k-1})$ for which the auxiliary state is a catalytic state can be found as follows:

(i) Fix one possible ordering of the Schmidt coefficients of $|\psi\rangle\otimes|\chi(\mathcal{P})\rangle$, and determine the set of all possible $\mathcal{P}$ that satisfies this ordering by solving the underlying nk linear inequalities. Hence, the set of $\mathcal{P}$ that correspond to a feasible fixed ordering of the eigenvalues of $|\psi\rangle\otimes|\chi(\mathcal{P})\rangle$, is a polyhedron (if an ordering is not feasible for any choice of $\mathcal{P}$, then the corresponding polyhedron is an empty set): the solutions of a set of linear inequalities defines a polyhedron. Also note that there are only a finite number of possible orderings of the eigenvectors of $|\psi\rangle\otimes|\chi(\mathcal{P})\rangle$, leading to a finite number of corresponding polyhedra, $\mathcal{O}_1^{\psi},\mathcal{O}_2^{\psi},\ldots,\mathcal{O}_L^{\psi}$. An accurate estimate of L can be obtained by viewing the counting problem as the number of possible ways k sorted lists, each of length n, can be merged to generate distinct sorted lists of length nk; an upper bound on it is $(nk)!$.

(ii) Similarly, compute the polyhedron for each ordering of the eigenvalues of $|\phi\rangle\otimes|\chi(\mathcal{P})\rangle$. Again, this yields at most $\mathcal{O}_1^{\phi},\mathcal{O}_2^{\phi},\ldots,\mathcal{O}_L^{\phi}$ polyhedra.

Now consider all possible polyhedra that are the intersections of pairs of nonempty order-preserving polyhedra defined above, i.e., $O_k=\mathcal{O}_i^{\psi}\cap\mathcal{O}_j^{\phi}$, $1\leqslant i,j\leqslant L$. The set of all points in any such polyhedron O_k that correspond to *catalytic states*, consists of those points in O_k that satisfy the underlying $nk-1$ majorization linear inequalities [see Eq. (1)], $\lambda_{\psi\otimes\chi(\mathcal{P})}<\lambda_{\psi\otimes\chi(\mathcal{P})}$. Hence, the catalytic states within O_k forms a polyhedron itself. Thus, each polyhedron representing values of $\mathcal{P}$ that correspond to catalytic states for the given pair $\{|\psi\rangle,|\phi\rangle\}$, can be viewed as the intersections of three different polyhedra: (i) the set of $\mathcal{P}$ corresponding to a fixed ordering of the Schmidt coefficients of $|\psi\rangle\otimes|\chi(\mathcal{P})\rangle$, (ii) the set of $\mathcal{P}$ corresponding to a fixed ordering of the Schmidt coefficients of $|\phi\rangle\otimes|\chi(\mathcal{P})\rangle$, and (iii) the set of all $\mathcal{P}$ that satisfy the majorization relations corresponding to the fixed orderings defined in (i) and (ii). We define such a polyhedron (which is the intersection of the preceding three polyhedra) as an *order preserving majorization polyhedron* (OPMP). ■

For catalytic states in any dimension $k\times k$, a typical OPMP S_i, can be represented by the extreme points (or vertices) of the underlying polyhedron: $S_i=\{\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_m\}$, where $\mathcal{P}_i\in\mathcal{R}^{k-1}$ and $E(|\chi(\mathcal{P}_1)\rangle)\geqslant E(|\chi(\mathcal{P}_2)\rangle)\geqslant\cdots\geqslant E(|\chi(\mathcal{P}_m)\rangle)$. For example, for $k=2$, one can represent each OPMP as an interval belonging to the segment $[\frac{1}{2},1]$, $S=[p_l,p_u]$, where $E(|\chi(p_l)\rangle)>E(|\chi(p_u)\rangle)$. By following the procedure outlined in the proof of the preceding lemma, it is fairly easy to construct all OPMPs for any given catalyzable incomparable pair, especially for small values of n and k. For instance, an OPMP for the states given by Eqs. (2) and (3) is $S_1=[\frac{10}{19},\frac{25}{38}]$. Another OPMP for the same pair but corresponding to a different ordering is $S_2=[\frac{13}{25},\frac{10}{19}]$.

The framework introduced in Lemma 1. shows that the set of all possible catalysts can be structured in terms of a *discrete and a finite* number of polyhedra, each of which has an efficient description (i.e., the corresponding vertices). Hence, our framework provides a *succinct necessary and sufficient condition* for determining whether a given pair of incomparable states is catalyzable or not, as captured in the following theorem.

Theorem 1. A given $n\times n$ incomparable pair of states is catalyzable if and only if there exists a nonempty OPMP in some $k\times k$.

Proof. Note that the computational problem for finding catalysts [i.e., given a pair of incomparable states in $n\times n$, does there exist a catalytic state in $k\times k$?] is in the class NP (nondeterministic polynomial) [13]: in order to provide a valid certificate for a ‘‘yes’’ instance of the problem, all one needs to do is to provide a candidate catalytic state $|\chi\rangle$, and one can verify in $O(nk)$ time whether $|\chi\rangle$ is indeed a catalytic state or not. Lemma 1 and Theorem 1 provide an $O([(nk)!]^2)$ algorithm not only to solve the ‘‘yes/no’’ version of the problem, but also to determine all the possible catalytic states. Whether the catalysis problem admits an efficient solution, or is an NP-complete problem, is left as an open problem. The preceding understanding of the structure of catalytic states can now be used to establish a connection between catalysis and supercatalysis and establish a sufficient condition for the latter. First, we introduce certain structures of the majorization relations. A *parameterized* majorization relationship, $\lambda_{\psi\otimes\chi(\mathcal{P})}<\lambda_{\phi\otimes\chi(\mathcal{P})}$, where $\mathcal{P}=(p_1,p_2,\ldots,p_{k-1})$, is said to be *strict* if there exists an OPMP of dimension $\geqslant 1$ (i.e., it is nonempty and is not a single point), such that there exists a point $\mathcal{P}_1$ in the OPMP where *all* the nontrivial $(nk)-1$, majorization inequalities [see Eq. (1)] are strict. We represent strict majorization as $\lambda_{\psi\otimes\chi(\mathcal{P})}\subset\lambda_{\phi\otimes\chi(\mathcal{P})}$. Moreover, a parameterized majorization relationship $\lambda_{\psi\otimes\chi(\mathcal{P})}<\lambda_{\phi\otimes\chi(\mathcal{P})}$ is said to be *semistrict* if there exists an OPMP of dimension $\geqslant 1$ (i.e., it includes at least a line segment), such that there exist a point $\mathcal{P}_1$ in the OPMP and a direction vector $\vec{d}\in\mathcal{R}^{k-1}$ such that $\mathcal{P}_1-\epsilon\vec{d}$ is also in the OPMP, and any equality relations in the majorization relationship at $\mathcal{P}_1$ holds even if $\mathcal{P}_1$ is replaced by $\mathcal{P}_1-\epsilon\vec{d}$ on the *right-hand side*; we refer to such equalities in the majorization relationships as *benign* [11]. Note that *strict majorization is a special case of the semistrict case*, and we represent semistrict majorization as $\lambda_{\psi\otimes\chi(\mathcal{P})}\subseteq\lambda_{\phi\otimes\chi(\mathcal{P})}$ [11]. Note also that since $E(|\chi(\mathcal{P})\rangle)$ is a concave function, then without loss of generality, we can assume that it *increases* along the direction $-\vec{d}\in\mathcal{R}^{k-1}$ (if not, then just reverse the sign of $\vec{d}$).

Theorem 2. Given an $n\times n$ catalyzable incomparable pair $\{|\psi\rangle,|\phi\rangle\}$ that admits catalysts in $k\times k$, *supercatalysis also occurs in* $k\times k$ for the given incomparable pair if $\lambda_{\psi\otimes\chi(\mathcal{P})}\subseteq\lambda_{\phi\otimes\chi(\mathcal{P})}$.

Proof. Since $\lambda_{\psi\otimes\chi(\mathcal{P})}\subseteq\lambda_{\phi\otimes\chi(\mathcal{P})}$, then it follows from the preceding definitions that there exist $\mathcal{P}_1,\vec{d}\in\mathcal{R}^{k-1}$, and an

$\epsilon > 0$ such that $\lambda_{\psi\otimes\chi(\mathcal{P}_1)} < \lambda_{\phi\otimes\chi(\mathcal{P}_1-\epsilon\vec{d})}$. The proof is direct, first pick a valid direction vector $\vec{d}$ and an ϵ small enough so that $\mathcal{P}_2 = \mathcal{P}_1 - \epsilon\vec{d}$ is still in the OPMP and all the majorization inequalities are still satisfied when $\mathcal{P}_2$ is used for the right-hand side of the majorization inequalities. Moreover, note that the entropy function increases along the direction $-\vec{d}$. Hence, to obtain supercatalysis, set $|\chi\rangle = |\chi(\mathcal{P}_1)\rangle$ as the initial entangled state and $|\omega\rangle = |\chi(\mathcal{P}_1 - \epsilon\vec{d})\rangle$ as the final auxiliary entangled state. ■

We next discuss the amount by which the entanglement of the auxiliary state can be enhanced by using the constructive procedure stated in Theorem 2. In other words, we would like to maximize the enhancement $\delta = E(\omega) - E(\chi)$, because by doing so the overall loss of entanglement in the transformation is minimized. In the procedure of Theorem 2, since both $|\chi\rangle$ and $|\omega\rangle$ belong to the same OPMP, say $S = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m\}$ (recall that the vertices of the OPMP are ordered in terms of decreasing entanglement), then the maximum enhancement $\delta \leqslant E(|\chi(\mathcal{P}_1)\rangle) - E(|\chi(\mathcal{P}_m)\rangle)$. Take for instance, one of the OPMP's for the states in Eqs. (2) and (3), $S_1 = [\frac{10}{19}, \frac{25}{38}]$. If we choose $|\omega\rangle = |\chi(\frac{10}{19})\rangle$ and $|\chi\rangle = |\chi(\frac{25}{38})\rangle$ then one can check that the transformation $|\psi\rangle \otimes |\chi\rangle \nrightarrow |\phi\rangle \otimes |\omega\rangle$ is not possible with certainty by LOCC. This shows that the preceding upper bound on the enhanced entanglement is not always attained. However, one can verify that the conditions of Theorem 2 are satisfied by S_1, and that one find two catalyst states in S_1 such that supercatalysis does indeed happen. Next, consider another OPMP for the same incomparable pair, $S_2 = [\frac{13}{25}, \frac{10}{19}]$. In this case, one can easily prove that the upper bound is indeed attained. It is clear that the amount of enhancement depends on the choice of OPMP. An optimal strategy would be to consider all possible OPMPs and to obtain the optimal pair that belongs to one particular OPMP for supercatalysis. This is, however, beyond the scope of this paper.

We now come to the question of *efficiency of supercatalysis*. The dimension of the auxiliary state $|\chi\rangle$, plays a crucial role in determining the complexity and efficiency of an entanglement assisted transformation. To reduce complexity and increase efficiency, it is necessary to keep the dimension of the borrowed entanglement at a minimum whenever possible. Theorem 2 provides sufficient conditions where catalysis leads to supercatalysis, *without increasing the dimension* of the auxiliary entangled states. However, we show next that there exist cases where catalysts exist in $k\times k$, but supercatalysis can never happen without increasing the dimension of the auxiliary states. Consider the following incomparable parent states in 5×5: $\psi = (0.4,0.3,0.2,0.05,0.05)$, and $\phi = (0.4,0.35,0.14,0.11,0)$. One can verify that this incomparable pair admits a catalyst $|\chi\rangle = (0.6,0.4)$. The following theorem, however, shows that the parent incomparable states *cannot participate* in any supercatalysis, without increasing the dimension of the entangled states to $\geqslant 3$.

Theorem 3. Let $\{|\psi\rangle, |\phi\rangle\}$ be an incomparable pair with eigenvalue vectors $\lambda_\psi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}, \lambda_\phi = \{\beta_1, \beta_2, \ldots, \beta_n\}$. If $\alpha_1 = \beta_1$ or $\alpha_n = \beta_n$ then supercatalysis is not possible with 2×2 auxiliary states. Moreover if $\alpha_1 = \beta_1$ and $\alpha_n = \beta_n$ then there are no 3×3 auxiliary states for supercatalysis.

Proof. Let there exist an auxiliary entangled state $|\chi\rangle$ such that $|\psi\rangle \otimes |\chi\rangle \to |\phi\rangle \otimes |\omega\rangle$ where $E(|\omega\rangle) > E(|\chi\rangle)$. Let $\lambda_\chi = \{p, 1-p\}, \lambda_\omega = \{q, 1-q\}$. Since $E(|\omega\rangle) > E(|\chi\rangle)$, therefore, $p > q$. Since $|\psi\rangle \otimes |\chi\rangle \to |\phi\rangle \otimes |\omega\rangle$, we have $\alpha_1 p \leqslant \beta_1 q$. Since $\alpha_1 = \beta_1$, therefore, $p \leqslant q$, which is a contradiction. Similar proof for the case when $\alpha_d = \beta_d$.

To prove the second part of the lemma assume there are 3×3 auxiliary states $|\chi\rangle$ and $|\omega\rangle$ such that $|\psi\rangle \otimes |\chi\rangle \to |\phi\rangle \otimes |\omega\rangle$ where $|\chi\rangle \to |\phi\rangle \otimes |\omega\rangle$. Let $\lambda_\chi = \{p_1, p_2, 1-p_1-p_2\}$ and $\lambda_\omega = \{q_1, q_2, 1-q_1-q_2\}$. We then have $\alpha_1 p_1 \leqslant \beta_1 q_1 \Rightarrow p_1 \leqslant q_1$ since $\alpha_1 = \beta_1$ and $1 - \alpha_n(1-p_1-p_2) \leqslant 1 - \beta_n(1-q_1-q_2) \Rightarrow p_1 + p_2 \leqslant q_1 + q_2$. Hence, $\lambda_\chi < \lambda_\omega$ and $|\chi\rangle \to |\omega\rangle \Rightarrow E(|\omega\rangle) < E(|\chi\rangle)$ (see Ref. [7]), which is a contradiction. ■

What happens if one cannot obtain auxiliary states for supercatalysis in the same dimension as the catalysts? Since the augmented pair $\{|\psi\rangle \otimes |\chi\rangle, |\phi\rangle \otimes |\chi\rangle$ is LOCC transformable, one can state the following result based on the results on recovery of entanglement in Ref. [11].

Theorem 4. Let $|\psi\rangle = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $|\phi\rangle = (\beta_1, \beta_2, \ldots, \beta_{n1})$, be an incomparable pair, where $\alpha_n \neq \beta_n$, and let the pair admit a $k\times k$ catalyst $|\chi\rangle$. Then the pair $\{|\psi\rangle, |\phi\rangle\}$ admits supercatalysts, with initial auxiliary state $|\chi'\rangle = |\chi\rangle \otimes |\chi_1\rangle$, and the final enhanced auxiliary state $|\omega'\rangle = |\chi\rangle \otimes |\omega_1\rangle$, where $|\chi_1\rangle$ and $|\omega_1\rangle$ are in dimension $m\times m$, $m \leqslant nk-1$ and $E(|\omega_1\rangle) > E(|\chi_1\rangle)$.

To summarize, we have shown the existence of entanglement assisted transformations that are more efficient than catalysis. In such transformations, called supercatalysis, the entanglement of the auxiliary state is enhanced at the end and, therefore, the net loss in entanglement is reduced. We obtained a set of sufficient conditions for supercatalysis to exist and explored several relationships between supercatalysis and catalysis. There are many open questions of interest, including: What are some of the necessary conditions for supercatalysis? Are the auxiliary states participating in a supercatalysis process also catalysts for the parent incomparable states? Is the existence of catalysis always sufficient to ensure supercatalysis? Are the problems of finding catalysts and supercatalysts for a given incomparable pair *NP* complete?

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