Exact Entanglement Cost of Multi-Qubit Bound Entangled States

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Abstract

We report the *exact* entanglement cost of a class of multiqubit bound entangled states, computed in the context of a universal model for multipartite state preparation. The exact amount of entanglement needed to prepare such states are determined by first obtaining lower bounds using a cut-set approach, and then providing explicit local protocols achieving the lower bound.

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I. INTRODUCTION

Quantum entanglement [1] has emerged as a new type of physical resource, and is the key ingredient for quantum information processing (QIP) tasks, including teleportation [2], super dense coding [3], and secure key distribution [4]. A physical resource (e.g., heat energy) is typically characterized by a measure or unit, so that a given system could be assigned a physically meaningful estimate of how much of the resource it contains (e.g., BTUs or calories). The purpose of defining such measures is operational and practical: for example, if we are given an amount of fuel with say 10,000 BTUs of recoverable heat energy, then one can derive a quantitative estimate of how large a space it can heat and what should be the efficiency of the conversion process. One could ask analogous questions for entanglement: is there a measure of entanglement such that one can use it to convey how many units of entanglement one can extract from a given state, or how many units would one require to prepare it. The critical importance of defining such relevant units of entanglement was recognized early on by the QIP community and a considerable amount of effort has been put into it [5, 6, 7]. Such efforts have met with only mixed success, and the case of multipartite entanglement has proven to be particularly difficult.

In the standard model for entanglement transformation, a state is shared by a set of spatially separated parties, and is manipulated via local operations and classical communications (LOCC). For the case of *bipartite* states, one can indeed define a physically informative unit referred to as the *ebit*, where one ebit stands for the entanglement content of a singlet state. Given a bipartite state, the *entanglement cost* of a state is the number of singlets required to prepare the state asymptotically, and *entanglement of distillation* is the amount of pure entanglement that can be extracted per copy, also in the asymptotic sense [5, 8, 9] can be suitably expressed in terms of ebits.

Multipartite quantum systems, on the other hand pose considerable difficulty in defining and computing a reasonable measure for entanglement. For instance, given a quantum state comprising of N subsystems, one can group the subsystems into $2 \leq M < N$, groups, where some of the subsystems can be considered to be a joint subsystem in a larger Hilbert space. *Entanglement* of each such M-partition is in general different, and indeed, a state which may be separable under certain partitions might be entangled in other partitions. However, for GHZ class of states, the reduced entropy as in pure bipartite states has been shown to be an appropriate entanglement measure [10]. Also a geometric measure of entanglement, originally proposed for bipartite systems, has been generalized for the case of multipartite entangled states [11, 12, 13]. The measure tries to estimate the distance in the Hilbert space from the closest separable states, and provides a lower bound for entanglement of formation [14]. This bound has been computed [17] for the unlockable bound entangled states of Smolin [15] and Dur [16]. As for the entanglement cost of multipartite states, there is no proposed general measure that can be reasonably defined and accurately estimated for a large enough class of states.

This letter provides an exact entanglement cost of preparing a class of multipartite boundentangled states recently introduced in Ref [19, 20]. The entanglement costs of these states are computed in the context of a universal model for multipartite state preparation: the minimum total bipartite entanglement required to prepare the given states via local operations and classical communications (LOCC) and starting from an initial state comprising only pairwise shared bipartite entanglements. The exact costs are determined by first computing lower bounds using a cut-set approach, and then providing explicit protocols for preparing the multipartite states (i.e., via LOCC and pairwise shared bipartite entanglements among the parties) that use the same total entanglement as the lower bounds.

There are several implications of the results in this letter worth noting: (i) To our knowledge, the exact entanglement cost of any mixed multipartite entangled states, be it distillable or bound entangled, has not been reported so far in the literature. These are first known exact entanglement cost of multipartite mixed states. (ii) For bipartite systems, the question whether the asymptotic entanglement cost per copy can become zero for a bound entangled state has been resolved recently [18]. The question however was open for multipartite states, and we answer it by showing that the entanglement cost of a multipartite boundentangled state does not approach zero in general in the asymptotic limit. (iii) It has been demonstrated that in bipartite systems, asymptotic manipulation is more efficient than single copy. Our result shows that in multipartite case, even if the state involved is a mixed one, asymptotic manipulation may not be more efficient than single copy.

II. A UNIVERSAL MODEL FOR COMPUTING THE ENTANGLEMENT COST OF MULTIPARTITE STATES

For bipartite systems, any state preparation involves pre-shared entanglement between the two parties which is then manipulated via LOCC to prepare the state in question. This can be suitably generalized for multipartite systems as well. A universal model for the preparation of multipartite states can be described as follows: The spatially separated parties start with pair-wise and independent bipartite entanglement. The parties then use LOCC to prepare the desired state. This universal model provides a unique means of computing entanglement cost for multipartite states: the sum of all the pairwise ebits used in the *optimal* preparation of the state. More precisely, for multi-qubit systems, let ρ be a N-qubit state to be prepared. Suppose now ρ_B be another N qubit state comprising only of pairwise bipartite entanglements having the form: $\rho_B = \otimes \prod_{i<j}^N \sigma_{ij}$, where, the index ij refers to the pair of parties (i, j) sharing the state σ_{ij} . Let the distillable entanglement between every pair of parties, (i, j) be e_{ij} measured in ebits. It is clear that there always exists a ρ_B such that ρ can be prepared from ρ_B via LOCC (in general under asymptotic manipulations), i.e., $\rho_B^{\otimes n} \to \rho^{\otimes m}$ where $\frac{m}{n} \to constant$ as $n \to \infty$. The entanglement cost, $E_C(\rho)$, is given as:

$$E_C(\rho) = \min\left(\sum_{i < j} e_{ij}\right) \ . \tag{1}$$

That is, the entanglement cost of a multipartite state is the sum of the bipartite entanglement between the parties minimized over all possible strategies for preparing the state. The above definition has a straightforward generalization for a general N party system where the i-thparty holds a quantum system of dimension d_i .

A cut-set (C-S) approach to Estimate lower bounds on E_C

While solving the above minimization problem is not an easy task in general, calculating non-trivial lower bounds is a lot more straightforward. We can use the well-known truism that distillable entanglement across a bipartite cut cannot increase under LOCC. If for example, in a two-party configuration of a multipartite state one can distill n ebits of entanglement between the two groups, then while preparing the entangled state in question, when all the relevant parties are spatially separated, one must have had spent at least n ebits of entanglement across the same cut. Thus, the sum of the pairwise bipartite entanglements crossing the cut must be at least n ebits. By computing distillable entanglement across all possible (or even a subset) of the bipartite cuts, one can always obtain a lower bound on the amount of entanglement that one needs to spend in preparing the state.

Remark 1: Note that since distillable entanglement can only increase in the asymptotic case, the lower bound on the cost is *independent of whether the distillable entanglements*, used in computing the bound, *correspond to the few-copy or the asymptotic case*. Hence, any lower bound using the cut-set approach is also a *lower bound* in an *asymptotic* sense.

Achieving the lower bound

The above two observations lead to the following strategy adopted in this work: (i) Compute a lower bound on the entanglement cost of a given multipartite state, based on the distillable entanglement across different possible partitions, and (ii) then, search for a strategy to prepare the given state using LOCC on another state which consists of only pairwise shared bipartite entanglement (i.e., ebits) such that the total bipartite entanglement equals the lower bound computed in step (i).

Remark 2: If we find a state consisting of pairwise shared ebits from which (*even if it is the single-copy case*) we can prepare the given multipartite state using LOCC, and the total bipartite entanglement equals the *lower bound* computed in step (i), then the bound from step (i) is also the *exact entanglement cost* of the given multipartite state. That is, *even though the state is prepared using a single copy* of a state comprising pairwise bipartite entanglement, one *cannot do any better using asymptotic manipulations*. This is because, the lower bound is already in the asymptotic sense (see Remark 1), and hence, if one could use less overall bipartite entanglement using asymptotic manipulations, then it will lead to contradiction.

We now use the above strategy to calculate the *exact entanglement cost* of a general class of *multipartite bound-entangled states*.

III. ENTANGLEMENT COST FOR MULTI-QUBIT BOUND ENTANGLED STATES

Recall that a multipartite quantum state is said to be bound entangled if there is no distillable entanglement between any subset as long as *all* the parties remain spatially separated from each other. If for such a state, entanglement can be distilled between two parties by bringing a subset of the other parties together, then the state is said to be an *activable bound entangled* (ABE) state. We now briefly describe a class of Bell-correlated ABE (BCABE) states, introduced in [19, 20]. First, however, we introduce few notations. The customary two qubit Bell states are defined as follows:

$$\left|\Phi^{\pm}\right\rangle = \frac{1}{\sqrt{2}} \left(\left|00\right\rangle \pm \left|11\right\rangle\right), \left|\Psi^{\pm}\right\rangle = \frac{1}{\sqrt{2}} \left(\left|01\right\rangle \pm \left|10\right\rangle\right)$$
(2)

Consider now a system comprising of $2N, N \geq 2$ qubits. Let $|p_i\rangle = |a_1^i a_2^i \dots a_{2N}^i\rangle$ where $a_1^i = 0$, and $a_j^i \in \{0, 1\}$, for all $j = 2, \dots, 2N$ such that there is an even number of 0s in the string $a_1^i a_2^i \dots a_{2N}^i$. Likewise, let $|q_i\rangle = |b_1^i b_2^i \dots b_{2N}^i\rangle$, where $b_1^i = 0$, and b_2^i, \dots, b_{2N}^i are either 0 or 1 with odd number of 0s in the string $b_1^i b_2^i \dots b_{2N}^i$. One can also define the states orthogonal to $|p_i\rangle$, $|q_i\rangle$ as: $|\overline{p_i}\rangle = \left|\overline{a_1^i a_2^i} \dots \overline{a_{2N}^i}\right\rangle$ and $|\overline{q_i}\rangle = \left|\overline{b_1^i b_2^i} \dots \overline{b_{2N}^i}\right\rangle$ where $\left\langle\overline{a_j^i}|a_j^i\right\rangle = 0 = \left\langle\overline{b_j^i}|b_j^i\rangle$, $\forall j = 1, \dots, 2N$ and $i = 1, \dots, 2^{2N-2}$. Note that the four sets of states, defined by $|p_i\rangle$'s, $|\overline{p_i}\rangle$'s, $|\overline{p_i}\rangle$, and $|\overline{q_i}\rangle$'s respectively, are non-overlapping and all have same cardinality, and they together span the complete Hilbert space of 2N qubit systems.

Now define the cat or GHZ basis:

$$\left|\Phi_{i}^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|p_{i}\right\rangle \pm \left|\overline{p_{i}}\right\rangle\right), i = 1, \dots, 2^{2N-2}$$

$$(3)$$

$$\left|\Psi_{i}^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|q_{i}\right\rangle \pm \left|\overline{q_{i}}\right\rangle\right), i = 1, ..., 2^{2N-2}$$

$$\tag{4}$$

We will use the notation [·] for pure state projector $|\cdot\rangle\langle\cdot|$. Let us now define the following 2N qubit density matrices:

$$\rho_{2N}^{\pm} = \frac{1}{2^{2N-2}} \sum_{i=1}^{2^{2N-2}} \left[\Phi_i^{\pm} \right], \sigma_{2N}^{\pm} = \frac{1}{2^{2N-2}} \sum_{i=1}^{2^{2N-2}} \left[\Psi_i^{\pm} \right]$$
(5)

In [19] an interesting recursive relation was derived relating bound entangled states of 2N-2 qubits with that of 2N qubits:

$$\rho_{2N}^{\pm} = \frac{1}{4} (\left[\Phi^{+}\right] \otimes \rho_{2N-2}^{\pm} + \left[\Phi^{-}\right] \otimes \rho_{2N-2}^{\mp} + \left[\Psi^{+}\right] \otimes \sigma_{2N-2}^{\pm} + \left[\Psi^{-}\right] \otimes \sigma_{2N-2}^{\mp}) \tag{6}$$

$$\sigma_{2N}^{\pm} = \frac{1}{4} (\left[\Psi^{+}\right] \otimes \rho_{2N-2}^{\pm} + \left[\Psi^{-}\right] \otimes \rho_{2N-2}^{\mp} + \left[\Phi^{+}\right] \otimes \sigma_{2N-2}^{\pm} + \left[\Phi^{-}\right] \otimes \sigma_{2N-2}^{\mp})$$
(7)

The class of states $\rho_{2N}^{\pm}, \sigma_{2N}^{\pm}$ have been shown to be activable bound entangled in Ref [19]. Let us just note that the states are bound entangled when all 2N parties are separated from each other. This is the configuration where the entanglement cost will be evaluated. Furthermore the set of states are connected to each other by local pauli operations on one qubit. Here we also note that if any 2N parties come together, they can do a joint measurement to discriminate the states $\{\rho_{2N-2}^+, \rho_{2N-2}^-, \sigma_{2N-2}^-, \sigma_{2N-2}^-\}$ (as they are mutually orthogonal, one would always be able to find such measurements). Then it follows from Eqs. (6) and (7), that they can create a maximally entangled state between the remaining two parties via LOCC. This implies there is one ebit of distillable entanglement across every 1: 2N - 1 bipartite partition.

In what follows, we show that N ebits are both necessary and sufficient to prepare a 2N $(N \ge 2)$ qubit ρ_{2N}^+ state.

A Lower Bound on Entanglement Cost

In our model, every state preparation starts from a quantum resource state of the form: $\rho_B = \bigotimes \prod_{i < j}^N \sigma_{ij}$, where, the index ij refers to the pair of parties (i, j) sharing the state σ_{ij} . We begin with 2N spatially separated nodes sharing such a resource state where $e_{ij} = e_{ji}$, $i, j \in \{A_1, A_2, A_3, ..., A_{2N}\}$ and $i \neq j$, be the bipartite distillable entanglement (measured in ebits) present between two parties A_i, A_j . In the state ρ_{2N}^+ , consider the 2N, 1: 2N - 1bipartite cuts like, $A_k : \{A_{i,i} \neq k\}$. Across each one of these cuts, one can distill one ebit of entanglement. Since LOCC can never increase distillable entanglement, or amount of entanglement spent across a cut in preparing the state should always be equal or more than the amount of distillable entanglement across that cut, then we must have for a cut like $A_k: \{A_{i,i} \neq k\}$,

$$\sum_{i,i\neq k} e_{A_k A_i} \ge 1 \tag{8}$$

We get one such inequality from each cut, corresponding to every party, and if we sum them up, then we get

$$\sum_{i,k,i\neq k} e_{A_k A_i} \ge 2N \tag{9}$$

Since, $e_{A_kA_i} = e_{A_iA_k}$, we have,

$$E = \sum_{k < i} e_{A_k A_i} \ge N,\tag{10}$$

where E in the unit of ebits is the total bi-partite entanglement shared between all the parties. One might be tempted to argue that the above bound has been derived from a single copy and hence, is not an asymptotic bound. However, as stated in *Remark 1*, any asymptotic manipulation can only increase the distillable entanglement of one ebit (as obtained from a single copy manipulation) across any 1:(2N-1) cut, and hence the lower bound in Eq. (10) is a valid lower bound.

A local protocol achieving the lower bound

We now give a protocol that utilizes N pairs of singlets and LOCC to prepare ρ_{2N}^+ . In fact, we show that a single copy of the original state, where N singlets are shared by N disjoint pairs, is enough to prepare a single copy of the BCABE states, and no asymptotic manipulation is necessary to achieve the lower-bound derived above. The proof that our protocol indeed works can be seen via induction. To begin with consider the state of four qubits, say A, B, C and D. The following state was first presented by Smolin [15] and corresponds to our class when N = 2.

$$\rho_{ABCD}^{+} = \frac{1}{4} ([\Phi^{+}]_{AB} \otimes [\Phi^{+}]_{CD} + [\Phi^{-}]_{AB} \otimes [\Phi^{-}]_{CD} + [\Psi^{+}]_{AB} \otimes [\Psi^{+}]_{CD} + [\Psi^{-}]_{AB} \otimes [\Psi^{-}]_{CD})$$
(11)

Let the pairs, (A, B), and (C, D), share a singlet each. A and C can classically communicate among themselves to prepare a state $|\Phi_i\rangle^{AA} \otimes |\Phi_i\rangle^{CC}$ randomly with equal probability. This can be done as follows: Assume that A and C, each of them possesses a Bell state generator. The generators however generate identical Bell states randomly based on a string of classical bits that can be established a priori. A and C then can each teleport one qubit of the correlated Bell states (keeping one qubit from each state to themselves) to B and D respectively using the shared singlets. This creates the state ρ_{ABCD}^+ . Suppose now we have two additional parties E and F and all three pairs (A, B), (C, D) and (E, F) share a singlet among thems and they would like to prepare the following six qubit state:

$$\rho_{ABCDEF}^{+} = \frac{1}{4} \left(\left[\Phi^{+} \right]_{EF} \otimes \rho_{ABCD}^{+} + \left[\Phi^{-} \right]_{EF} \otimes \rho_{ABCD}^{-} + \left[\Psi^{+} \right]_{EF} \otimes \sigma_{ABCD}^{+} \right) + \left[\Psi^{-} \right]_{EF} \otimes \sigma_{ABCD}^{-} \right)$$

$$(12)$$

Let us further note that the three other bound entangled states belonging to the same class can be obtained by applying an appropriate local pauli rotation on any one of the qubits. To begin with, A, B, C and D will prepare a four qubit state as described before. Suppose E has a Bell state generator and A has a machine that can apply a Pauli rotation on the qubit. Furthermore, the machines share a common random two bit classical string. Based on that classical string E's machine generates a Bell state and accordingly A's machine applies a Pauli rotation on the qubit of A. Once E teleports the qubit via the singlet shared with F, the six party state is produced among them.

It is obvious that the above strategy can be inductively extended to any number of parties 2N, for any N greater than three.

IV. DISCUSSIONS

As argued before. this result shows an interesting feature, i.e., to prepare a mixed state in a multiparty setting even by an asymptotic manipulation one may not do better than the single copy preparation. This is in contrast with bipartite entanglement manipulation where a mixed state preparation is necessarily more efficient asymptotically.

In Ref. [21] entanglement of creation was introduced as the number of qubits per copy exchanged between the parties to prepare the entangled state optimally. It was also shown that entanglement of creation is equal to entanglement of formation [14] for bipartite systems. For the bound entangled states considered in the work, let us now point out the equivalence of our concept of entanglement cost with that of entanglement of creation. First note that the distillable entanglement across any cut, as measured in ebits, is always a lower bound on the number of qubits that need to be exchanged across the same cut. Hence, the lower bounds derived here are also lower bounds for the entanglement of creation. Next, the pairwise entanglements we used in our constructive protocols are all singlets. A singlet can be always established by sending a qubit. Hence, the states can be prepared by exchanging the same number of qubits as the number of singlets used in our preparations. Since the number of singlets match the lower bounds, the actual number of qubits required to prepare the states also equal the lower bounds. Thus, the exact entanglement of creation of our 2N-party state is also N.

While preparing a single copy of the state, we showed how it can be done by using singlets shared between N pairs. This of course not the only local way to prepare such state. Take for instance the four qubit Smolin unlockable bound entangled state [15] which belongs to our class of states when N = 2. Instead of providing two disjoint pairs with two singlets, one can think of a square configuration where every edge has distillable entanglement equal to 0.5. In such a distribution, the Smolin state, can be manufactured with an efficiency of 2 ebits per copy only asymptotically. A similar efficiency can also be achieved providing every pair with states having distillable entanglement equal to 1/3. One should note that one can now in principle assign states with varying distillable entanglement between the parties but such a distribution would necessarily be inefficient in the sense the cost of preparation per copy would go up. Let us emphasize that only by providing singlets between the pairs one can achieve the optimal value for a single copy preparation.

Our approach has some obvious weaknesses. It relies heavily on the knowledge of exact distillable entanglement across all bipartite partitions of the multipartite state which are in general very difficult to compute. Considering an extreme case where our approach fails is to compute the entanglement cost of bound entangled states that are not activable. Our partitioning argument does not work because across every partition the state has zero distillable entanglement. However in many cases it can be computed like our's for example and in such situations it might be able to provide a good lower bound.

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