PHYSICAL REVIEW A, VOLUME 65, 040303(R)

Partial recovery of entanglement in bipartite-entanglement transformations

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(Received 11 June 2001; published 10 April 2002)

Any *deterministic* bipartite-entanglement transformation involving finite copies of pure states and carried out using local operations and classical communication (LOCC) results in a net loss of entanglement. We show that for almost all such transformations, partial recovery of lost entanglement is achievable by using 2×2 auxiliary entangled states, no matter how large the dimensions of the parent states are. For the rest of the special cases of deterministic LOCC transformations, we show that the dimension of the auxiliary entangled state depends on the presence of equalities in the majorization relations of the parent states. We show that genuine recovery is still possible using auxiliary states in dimensions less than that of the parent states for *all* patterns of majorization relations except only one special case.

DOI: 10.1103/PhysRevA.65.040303 PACS number(s): 03.67.−a, 03.65.−w

Entanglement, shared among spatially separated parties, is a critical resource that enables efficient implementations of several quantum-information processing [1,2] and distributed-computation [3] tasks. To better exploit the power of entanglement, considerable effort has been put into understanding its transformation properties [4–7] and characterizing transformations allowed under local operations and classical communication (LOCC). A central question is what happens to the overall entanglement during transformations? In the asymptotic limit involving infinite number of copies of pure states, entanglement can be concentrated and diluted with unit efficiency [4]. This remarkable asymptotic "nondissipative" property, however, does not hold in the finite copy regime, where the process becomes inherently "dissipative," and a local *deterministic* conversion between two pure entangled states (which are not locally unitarily related), *always* results in a net loss of entanglement [5].

It is of fundamental importance to devise local strategies to *recover* the lost entanglement in an entanglement manipulation. Such recovery strategies would require *collective* manipulations with ancillary resources. That is, let $|\psi\rangle=\Sigma_{i=1}^{n}\sqrt{\alpha_i}|i\rangle|i\rangle$ and $|\varphi\rangle=\Sigma_{i=1}^{n}\sqrt{\beta_i}|i\rangle|i\rangle$ be, respectively, the source and target states in $n\times n$ such that $|\psi\rangle\rightarrow|\varphi\rangle$ under LOCC with certainty. Then the amount of entanglement lost in such a transformation is $E(|\psi\rangle)-E(|\varphi\rangle)$ (where E is the entropy of entanglement), and we say that there is a *partial recovery* of the lost entanglement if there exist entangled states $|\chi\rangle$, $|\omega\rangle$ in $k\times k$, such that $|\psi\rangle\otimes|\chi\rangle\rightarrow|\varphi\rangle\otimes|\omega\rangle$ with certainty under LOCC, and $E(|\omega\rangle)>E(|\chi\rangle)$. Since the overall transformation involving the auxiliary states is dissipative, the recovered entanglement, $E(|\omega\rangle)-E(|\chi\rangle)$, is always less than or equal to the initial amount of lost entanglement, $E(|\psi\rangle)-E(|\varphi\rangle)$. In order to minimize the use of ancillary resources and to reduce the complexity of the collective operations, we consider a partial recovery of entanglement process to be *efficient*, if the dimension of the auxiliary states, k is the minimum required for the recovery process to happen. Moreover, in order for the partial recovery process to be *genuine*, we require the dimension of the auxiliary states to be smaller than that of the parent states (i.e., $k<n$), since otherwise, if $k=n$ then one can have a complete recovery of lost entanglement by a trivial choice $|\chi\rangle=|\varphi\rangle$ and $|\omega\rangle=|\psi\rangle$.

A first step toward achieving partial recovery of entanglement has recently been taken in [8] for the special case of $n=2$. This result is of limited interest only, since the auxiliary pure states are necessarily of the same dimension as the parent states (i.e., $k=n=2$), one can always have a complete recovery of lost entanglement by a trivial choice of the auxiliary states. However, [8] presents nontrivial selections of auxiliary states (i.e., $|\chi\rangle\neq|\varphi\rangle$) that lead to partial recovery of entanglement.

We prove that genuine and efficient partial recovery of entanglement is *always* possible for almost all bipartite-entanglement transformations in *any* finite dimension (i.e., for any $n>2$). Moreover, for almost all comparable pairs, such partial recovery is achievable by using auxiliary states of minimum possible dimension, i.e., $k=2$, no matter how large the dimensions of the parent states are. For the rest of the special cases of comparable parent states, we show that the dimension of the auxiliary entangled state depends on the structure of the majorization relations of the parent states, where the presence of equalities in the majorization relations either in isolation or in blocks determine the dimension of the auxiliary entanglement.

Recall that our parent bipartite pure states are represented as $|\psi\rangle=\Sigma_{i=1}^{n}\sqrt{\alpha_i}|i\rangle|i\rangle$ and $|\varphi\rangle=\Sigma_{i=1}^{n}\sqrt{\beta_i}|i\rangle|i\rangle$, where $\alpha_1\geqslant\alpha_2\geqslant\cdots\geqslant\alpha_n$ and $\beta_1\geqslant\beta_2\geqslant\cdots\geqslant\beta_n$, are the Schmidt coefficients. Define the vector of the eigenvalues of the reduced density matrices as $\lambda_\psi\equiv(\alpha_1,\ldots,\alpha_n)$ and $\lambda_\varphi\equiv(\beta_1,\ldots,\beta_n)$. Since our parent states are comparable, i.e.,

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$|\psi\rangle \to |\varphi\rangle$ with probability one under LOCC, it follows from [5] that λ_ψ is *majorized* by λ_φ (denoted as $\lambda_\psi \prec \lambda_\varphi$), i.e.,

$$\sum_{i=1}^{m} \alpha_i \leqslant \sum_{i=1}^{m} \beta_i \quad \text{for every } m=1,\ldots,n-1. \tag{1}$$

Note that both sides equal one for $m=n$.

First we consider the case where λ_ψ is *strictly majorized* by λ_φ, i.e., all the inequalities of the majorization conditions (1) are *strict*, and show that recovery with an auxiliary entangled state in 2×2 is always possible. We represent strict majorization as $\lambda_\psi \lhd \lambda_\varphi$. Note that for a randomly picked pair of comparable states, the majorization inequalities are strict with probability one. This guarantees that the case where all the majorization inequalities are strict covers *almost all* possible comparable pairs. We first illustrate the basic idea involved in the proof with a simple example.

Example. Consider the states $|\psi\rangle$ and $|\varphi\rangle$ with $\lambda_\psi=(0.4,0.3,0.2,0.1)$, and $\lambda_\varphi=(0.5,0.3,0.2,0)$. Then $\lambda_\psi \lhd \lambda_\varphi$. Note that since $|\psi\rangle \to |\varphi\rangle$, then for all 2×2 states $|\chi(p)\rangle$, where $\lambda_{\chi(p)}=(p,1-p)$ and $p\in(0.5,1)$, we have $|\psi\rangle \otimes |\chi(p)\rangle \to |\varphi\rangle \otimes |\chi(p)\rangle$. One can verify that for $p=0.8$ we have $\lambda_{\psi\otimes\chi(p)} \lhd \lambda_{\varphi\otimes\chi(p)}$, moreover, for any small perturbation around $p=0.8$, the ordering of the Schmidt coefficients of $|\varphi\rangle \otimes |\chi(p)\rangle$ is preserved. In particular, one can verify that the relation $\lambda_{\psi\otimes\chi(p)} \prec \lambda_{\varphi\otimes\chi(p-\varepsilon)}$ holds if $0<\varepsilon<0.08$. Thus, we can choose $|\chi\rangle = |\chi(0.8)\rangle$, and $|\omega\rangle=|\chi(0.73)\rangle$. Then $|\psi\rangle\otimes|\chi\rangle \to |\varphi\rangle\otimes|\omega\rangle$, where $E(|\omega\rangle)>E(|\chi\rangle)$. While in this example we directly provided a value of p around which a perturbation leads to partial recovery, the proof of the following theorem shows that such a p always exists, and outlines how one can find such a p systematically. ∎

Theorem 1. If $\lambda_\psi \lhd \lambda_\varphi$ then there are 2×2 states $|\chi\rangle$ and $|\omega\rangle$ such that $|\psi\rangle\otimes|\chi\rangle \to |\varphi\rangle\otimes|\omega\rangle$ and $E(|\omega\rangle)>E(|\chi\rangle)$.

Proof. Let $|\chi(p)\rangle$ be a 2×2 state with $\lambda_{\chi(p)}=(p,1-p)$, and $\frac{1}{2}<p<1$. Note that, in general, if $|\psi_1\rangle \to |\varphi_1\rangle$ and $|\psi_2\rangle \to |\varphi_2\rangle$ then $|\psi_1\rangle\otimes|\psi_2\rangle \to |\varphi_1\rangle\otimes|\varphi_2\rangle$. Therefore, for all values of $p\in(\frac{1}{2},1)$, $|\psi\rangle\otimes|\chi(p)\rangle \to |\varphi\rangle\otimes|\chi(p)\rangle$. The choice of p determines the orderings of the Schmidt coefficients of $|\psi\rangle\otimes|\chi(p)\rangle$ and $|\varphi\rangle\otimes|\chi(p)\rangle$, and hence the inequalities in $\lambda_{\psi\otimes\chi(p)} \prec \lambda_{\varphi\otimes\chi(p)}$. Conversely, one can think in terms of the orderings of the Schmidt coefficients. There is only a finite number (in fact, at most $n!$) of possible individual orderings of the coefficients of $|\psi\rangle\otimes|\chi(p)\rangle$ and $|\varphi\rangle\otimes|\chi(p)\rangle$. For each such ordering of the coefficients of $|\varphi\rangle\otimes|\chi(p)\rangle$, one can determine its feasible set: values of $p\in(\frac{1}{2},1)$ for which the ordering is valid. Each nonempty feasible set corresponds to the solution of a set of linear inequalities, and hence, is a union of intervals and discrete points in $(\frac{1}{2},1)$. Moreover, the union of the feasible sets of all possible orderings of the coefficients is the interval $(\frac{1}{2},1)$. Since the union of this *finite* set of intervals and discrete points is $(\frac{1}{2},1)$, hence, it follows from simple measure-theoretic arguments that there exists at least one ordering, where the corresponding feasible set F includes intervals of nonzero lengths of the form (a,b), where $\frac{1}{2}<a<b<1$.

Next, let us restrict p to belong to such a nontrivial feasible set F. We next show that $\lambda_{\psi\otimes\chi(p)} \lhd \lambda_{\varphi\otimes\chi(p)}$ for all values of $p\in F$, *except* at most $2n-1$ discrete values. Hence, the set of points p where the majorization inequalities are strict and the ordering of Schmidt coefficients is preserved is of nonzero measure, i.e., it includes intervals. If in the majorization relationship of $\lambda_{\psi\otimes\chi(p)} \prec \lambda_{\varphi\otimes\chi(p)}$ one of the inequalities (among the $2n-1$ nontrivial inequalities) is an equality, then we must have

$$p\sum_{j=1}^{x}\alpha_j+(1-p)\sum_{j=1}^{y}\alpha_j=p\sum_{j=1}^{s}\beta_j+(1-p)\sum_{j=1}^{t}\beta_j, \tag{2}$$

where $x+y=s+t$, $x\geqslant y$, and $s\geqslant t$. Equivalently,

$$\left(\sum_{j=1}^{x}\alpha_j-\sum_{j=1}^{y}\alpha_j-\sum_{j=1}^{s}\beta_j+\sum_{j=1}^{t}\beta_j\right)p=\sum_{j=1}^{t}\beta_j-\sum_{j=1}^{y}\alpha_j. \tag{3}$$

There are two cases now: (i) Eq. (3) determines a value of p, and (ii) Eq. (3) is an equivalence, and hence, does not determine a value for p . We show that case (ii) is impossible, Eq. (3) does not determine a value for p, if and only if $\Sigma_{j=1}^{x}\alpha_j=\Sigma_{j=1}^{s}\beta_j$ and $\Sigma_{j=1}^{y}\alpha_j=\Sigma_{j=1}^{t}\beta_j$. Since $\lambda_\psi \lhd \lambda_\varphi$, it follows that $x>s$ and $y>t$. This *contradicts* the condition $x+y=s+t$. Hence, every potential equality in the majorization relationship $\lambda_{\psi\otimes\chi(p)} \prec \lambda_{\varphi\otimes\chi(p)}$ corresponds to a fixed value for p. Since, there are at most $2n-1$ such nontrivial equalities, there are at most $2n-1$ values for p for which $\lambda_{\psi\otimes\chi(p)} \prec \lambda_{\varphi\otimes\chi(p)}$ is not strict.

Hence, there exist a $p\in F\subseteq(\frac{1}{2},1)$ and an $0<\varepsilon<\frac{1}{2}$ such that $\lambda_{\psi\otimes\chi(p)} \prec \lambda_{\varphi\otimes\chi(p-\varepsilon)}$. The proof is completed by setting $|\chi\rangle=|\chi(p)\rangle$ and $|\omega\rangle=|\chi(p-\varepsilon)\rangle$. ∎

What happens if λ_ψ is *not* strictly majorized by λ_φ? We first define $\Delta_{\psi,\varphi}$ as the set of all indices m such that the relation (1) is an equality: $\Delta_{\psi,\varphi}=\{m\colon 1\leqslant m\leqslant n-1,\ \Sigma_{i=1}^{m}\alpha_i=\Sigma_{i=1}^{m}\beta_i\}$. Note that $1\in\Delta_{\psi,\varphi}$ is equivalent to the case $\alpha_1=\beta_1$ and $n-1\in\Delta_{\psi,\varphi}$ is equivalent to the case $\alpha_n=\beta_n$.

We first show that even in the presence of many patterns of equalities in the majorization relationship of the parent states recovery is still possible using only 2×2 auxiliary states.

Theorem 2. Suppose that $1\notin\Delta_{\psi,\varphi}$ (i.e., $\alpha_1\neq\beta_1$), $n-1\notin\Delta_{\psi,\varphi}$ (i.e., $\alpha_n\neq\beta_n$), and if $j\in\Delta_{\psi,\varphi}$ then $j+1\notin\Delta_{\psi,\varphi}$ (i.e., there are no consecutive equalities in the majorization). Then there are 2×2 states $|\chi\rangle$ and $|\omega\rangle$ such that $|\psi\rangle\otimes|\chi\rangle \to |\varphi\rangle\otimes|\omega\rangle$ and $E(|\omega\rangle)>E(|\chi\rangle)$.

Proof. We first show that there exists a nonempty interval $I=(\frac{1}{2},a)$ $(1>a>\frac{1}{2})$, such that each inequality in the majorization relationship $\lambda_{\psi\otimes\chi(p)} \prec \lambda_{\varphi\otimes\chi(p)}$ is either (i) a *benign* identity for all $p\in I$, that is, the equality holds even if on the right-hand side p is perturbed to $p-\varepsilon$, for any $\varepsilon>0$, or (ii) is a *strict* inequality, for all $p\in I$, *except for* at most $2n-1$ discrete values. Such a majorization, where each inequality is either strict or a benign identity, is represented as $\lambda_{\psi\otimes\chi(p)} \trianglelefteq \lambda_{\varphi\otimes\chi(p)}$. One can then use simple measure-theoretic arguments, as introduced in the proof of Theorem 1, and show that there exists an ordering of the Schmidt coefficients of $|\varphi\rangle\otimes|\chi(p)\rangle$ such that $F\cap I$ has a nonzero measure (i.e., includes intervals), where F is the feasible set for the given ordering. The two above results then show that there exists a $p\in F\cap I$ such that $\lambda_{\psi\otimes\chi(p)}\underline{\lhd}\lambda_{\varphi\otimes\chi(p)}$, and in a neighborhood around p the ordering of the Schmidt coefficients of $|\varphi\rangle\otimes|\chi(p)\rangle$ is preserved. Hence, there is an $0<\varepsilon<\frac{1}{2}$, such that $\lambda_{\psi\otimes\chi(p)}\prec\lambda_{\varphi\otimes\chi(p-\varepsilon)}$. The proof can then be completed by setting $|\chi\rangle=|\chi(p)\rangle$ and $|\omega\rangle=|\chi(p-\varepsilon)\rangle$. We now present a construction of such a set I.

First consider the case where there are only two equalities, i.e., $\Delta_{\psi,\varphi}=\{k_1,k_2\}$, where $1<k_1<k_2<n-1$ and $k_2-k_1\geqslant 2$. Since $\alpha_1\neq\beta_1$, it cannot be the case that *both* $\alpha_1=\alpha_{k_1}$ and $\beta_1=\beta_{k_1}$: if it is true then the k_1^{th} inequality in the majorization is also strict and $k_1\notin\Delta_{\psi,\varphi}$, which contradicts our assumption. Hence, $\alpha_1>\alpha_{k_1}$ or $\beta_1>\beta_{k_1}$ or both. Similarly, one can argue that (i) since $k_1\in\Delta_{\psi,\varphi}$, and $k_1+1\notin\Delta_{\psi,\varphi}$, both $\alpha_{k_1+1}=\alpha_{k_2}$ and $\beta_{k_1+1}=\beta_{k_2}$ cannot be true, and (ii) since $k_2\in\Delta_{\psi,\varphi}$, and $k_1+1\notin\Delta_{\psi,\varphi}$, both $\alpha_{k_2+1}=\alpha_n$ and $\beta_{k_2+1}=\beta_n$ cannot be true. Now set $I=(\frac{1}{2},a)$, where

$$a=\min\left\{q_1\frac{\alpha_1}{\alpha_1+\alpha_{k_1}},q_2\frac{\beta_1}{\beta_1+\beta_{k_1}},q_3\frac{\alpha_{k_1+1}}{\alpha_{k_1+1}+\alpha_{k_2}},\right.$$
$$\left.q_4\frac{\beta_{k_1+1}}{\beta_{k_1+1}+\beta_{k_2}},q_5\frac{\alpha_{k_2+1}}{\alpha_{k_2+1}+\alpha_n},q_6\frac{\beta_{k_2+1}}{\beta_{k_2+1}+\beta_n}\right\},\qquad(4)$$

and $q_i=2$, if its accompanying multiplicative term equals $\frac{1}{2}$, otherwise $q_i=1$. Thus, if $\alpha_1=\alpha_{k_1}$, then $q_1=2$ and the first term, $q_1[\alpha_1/(\alpha_1+\alpha_{k_1})]=1$, plays no role in determining the value of a, otherwise, if $\alpha_1>\alpha_{k_1}$, then $q_1=1$ and the first term, $\frac{1}{2}<q_1[\alpha_1/(\alpha_1+\alpha_{k_1})]<1$, can potentially determine a. *By construction*, $\frac{1}{2}<a<1$, and hence, I is nonempty. The motivation of defining a as above is that by restricting $p\in(\frac{1}{2},a)$, it enforces a *partial ordering* of the Schmidt coefficients of $|\varphi\rangle\otimes|\chi(p)\rangle$ and $|\psi\rangle\otimes|\chi(p)\rangle$. For example, if $\beta_1>\beta_{k_1}$ then from Eq. (4) it follows that $p\beta_{k_1}<(1-p)\beta_1$, and hence, in the ordering of the Schmidt coefficients of $|\varphi\rangle\otimes|\chi(p)\rangle$, $(1-p)\beta_1$ will appear before $p\beta_{k_1}$.

Next, we show $\lambda_{\psi\otimes\chi(p)}\underline{\lhd}\lambda_{\varphi\otimes\chi(p)}$ for all $p\in F\cap I$, *except* at most $2n-1$ discrete values. In the majorization relationship of $\lambda_{\psi\otimes\chi(p)}\prec\lambda_{\varphi\otimes\chi(p)}$ let one of the inequalities (among the $2n-1$ nontrivial inequalities) be an equality. Then following arguments similar to those used in the proof of Theorem 1 and using the partial ordering of Schmidt coefficients enforced by the selection of I [see Eq. (4)], we show that either (i) the equality determines a value of p (hence, there are at most $2n-1$ discrete values of p where any such equality can exist), or (ii) the equality is a *benign* identity with one of the following forms:

$$p\sum_{i=1}^{k_j}\alpha_i+(1-p)\sum_{i=1}^{k_j}\alpha_i=p\sum_{i=1}^{k_j}\beta_i+(1-p)\sum_{i=1}^{k_j}\beta_i,\qquad(5)$$

where $j\in\{1,2\}$. The reason identities as in Eq. (5) are benign for our purposes is that when p is substituted by $p-\varepsilon$ on the right-hand side, then the identity still remains an equality. To prove the above claim, consider an equality in the majorization relationship, which as discussed in the proof of Theorem 1 [see Eq. (3)], can be written as

$$\left(\sum_{j=1}^{x}\alpha_j-\sum_{j=1}^{y}\alpha_j-\sum_{j=1}^{s}\beta_j+\sum_{j=1}^{t}\beta_j\right)p=\sum_{j=1}^{t}\beta_j-\sum_{j=1}^{y}\alpha_j,\qquad(6)$$

where $x+y=s+t$, $x\geqslant y$, and $s\geqslant t$. Equation (6) is an equivalence if and only if the following two conditions are simultaneously satisfied. (i) $\Sigma_{j=1}^{t}\beta_j=\Sigma_{j=1}^{y}\alpha_j$, which is true only if $t=y\in\{0,k_1,k_2\}$, or if $y>t$; and (ii) $\Sigma_{j=1}^{x}\alpha_j=\Sigma_{j=1}^{t}\beta_j$, which is true only if $x=s\in\{k_1,k_2,n\}$, or if $x>s$. The benign identity cases occur if $x=y=s=t=k_j$, $j\in\{1,2\}$. Let us then consider all the other potentially feasible cases and show that they are all *impossible*: (i) if ($y>t$ and $x\geqslant s$) or ($y\geqslant t$ and $x>s$) then we reach the contradiction that $x+y>s+t$; (ii) if $(y=t=0)$ and $(x=s\in\{k_1,k_2,n\})$, this implies that $(1-p)\alpha_1<p\alpha_{k_1}$ and $(1-p)\beta_1<p\beta_{k_1}$, which contradicts the fact that $p\in(\frac{1}{2},a)$ [see Eq. (4)]; (iii) if $(y=t=k_1)$ and $(x=s\in\{k_2,n\})$, this implies that $(1-p)\alpha_{k_1+1}<p\alpha_{k_2}$ and $(1-p)\beta_{k_1+1}<p\beta_{k_2}$, which again contradicts the construction introduced in Eq. (4); (iv) if $(y=t=k_2)$ and $(x=s=n)$, this implies that $(1-p)\alpha_{k_2+1}<p\alpha_n$ and $(1-p)\beta_{k_2+1}<p\beta_n$, which again contradicts the construction introduced in Eq. (4).

In general, when $\Delta_{\psi,\varphi}=\{k_1,k_2,\ldots,k_\ell\}$, where $1<k_1<\cdots<k_\ell<n-1$ and $k_{i+1}-k_i\geqslant 2$, then one can show the above results for $I=(\frac{1}{2},a)$, where

$$a=\min\left\{q_1\frac{\alpha_1}{\alpha_1+\alpha_{k_1}},q_2\frac{\beta_1}{\beta_1+\beta_{k_1}},\ldots,\right.$$
$$q_{2\ell-1}\frac{\alpha_{k_{\ell-1}+1}}{\alpha_{k_{\ell-1}+1}+\alpha_{k_\ell}},q_{2\ell}\frac{\beta_{k_{\ell-1}+1}}{\beta_{k_{\ell-1}+1}+\beta_{k_\ell}},q_{2\ell+1}\frac{\alpha_{k_\ell+1}}{\alpha_{k_\ell+1}+\alpha_n},$$
$$\left.q_{2(\ell+1)}\frac{\beta_{k_\ell+1}}{\beta_{k_\ell+1}+\beta_n}\right\},$$

and q_i's are chosen as in Eq. (4). ■

We next show that for certain equality patterns in the majorization relation, partial recovery with the help of 2×2 states (or even 3×3 states) is *not always possible.*

Lemma 1. If $\alpha_1=\beta_1$ or $\alpha_n=\beta_n$, then recovery is not possible with 2×2 auxiliary states. Also, if both relations $\alpha_1=\beta_1$ and $\alpha_n=\beta_n$ hold then there is no recovery even with 3×3 auxiliary states.

Proof. First assume that $\alpha_1=\beta_1$ or $\alpha_n=\beta_n$. Suppose, by contradiction, there are 2×2 states $|\chi\rangle$ and $|\omega\rangle$ such that $|\psi\rangle\otimes|\chi\rangle\rightarrow|\varphi\rangle\otimes|\omega\rangle$ and $E(|\omega\rangle)>E(|\chi\rangle)$. Let $\lambda_\chi=(p,1-p)$ and $\lambda_\omega=(q,1-q)$ be the vector of eigenvectors of $|\chi\rangle$ and $|\omega\rangle$ with $p,q>\frac{1}{2}$. The condition $E(|\omega\rangle)>E(|\chi\rangle)$ implies that $q<p$. The relation $\lambda_{\psi\otimes\chi}\prec\lambda_{\varphi\otimes\omega}$ implies that $\alpha_1 p$

$\leqslant \beta_1 q$ and $1-\alpha_n(1-p) \leqslant 1-\beta_n(1-q)$. So if $\alpha_1=\beta_1$ or $\alpha_n=\beta_n$ then $p \leqslant q$ and $E(|\omega\rangle) \leqslant E(|\chi\rangle)$, which is a contradiction.

The proof for the case, where $\alpha_1=\beta_1$ and $\alpha_n=\beta_n$, is similar to the above case: assume that there are 3×3 recovery states $|\chi\rangle$ and $|\omega\rangle$ [hence, $E(|\omega\rangle) > E(|\chi\rangle)$] with eigenvalue vectors $\lambda_\chi=(p,q,1-p-q)$ and $\lambda_\omega=(p',q',1-p'-q')$ with $p \geqslant q \geqslant 1-p-q$ and $p' \geqslant q' \geqslant 1-p'-q'$. Then it will follow that $|\chi\rangle \rightarrow |\omega\rangle$, which implies the contradiction that $E(|\omega\rangle) \leqslant E(|\chi\rangle)$ (see [5]). ■

The following theorem shows that if $\alpha_1=\beta_1$ then there indeed exist 3×3 auxiliary states for partial recovery.

Theorem 3. If $\Delta_{\psi,\varphi}=\{1\}$ then there are 3×3 states $|\chi\rangle$ and $|\omega\rangle$ such that $|\psi\rangle\otimes|\chi\rangle \rightarrow |\varphi\rangle\otimes|\omega\rangle$ and $E(|\omega\rangle) > E(|\chi\rangle)$.

Proof. Let $|\chi(p,q)\rangle$ be a 3×3 state with $\lambda_{\chi(p,q)}=(p,q,1-p-q)$, where $p \geqslant q \geqslant 1-p-q \geqslant 0$. The goal is to find a state $|\omega\rangle$ of the form $|\chi(p,q-\varepsilon)\rangle$, for some $\varepsilon>0$, such that $\lambda_{\psi\otimes\chi(p,q)} \prec \lambda_{\varphi\otimes\chi(p,q-\varepsilon)}$. Our approach is similar to that introduced in the proof of Theorem 2: we construct a region $R=\{(p,q) | p \geqslant q \geqslant 1-p-q \geqslant 0\}$, with nonzero area such that $\lambda_{\psi\otimes\chi(p,q)} \underline{\prec} \lambda_{\varphi\otimes\chi(p,q)}$ for almost all $(p,q)\in R$, and the set of points where it is violated has *measure zero*. Here, an identity is considered to be *benign* if the equality holds when on the right-hand side (p,q) is perturbed to $(p,q-\varepsilon)$. Then, measure-theoretic arguments will guarantee that there is an $\varepsilon>0$, such that $\lambda_{\psi\otimes\chi(p,q)} \prec \lambda_{\varphi\otimes\chi(p,q-\varepsilon)}$.

In order to construct R, we note that since $2\notin\Delta_{\psi,\varphi}$, $\alpha_1>\alpha_2$. Also note that if $\alpha_2=\alpha_n$ then $\beta_2>\beta_n$. Therefore, we have to consider one of two cases (i) $\alpha_1>\alpha_2>\alpha_n$ and (ii) $\alpha_1>\alpha_2=\alpha_n$ and $\beta_2>\beta_n$. To define R, we choose the parameters p and q such that $p \geqslant q \geqslant 1-p-q \geqslant 0$ and they satisfy the following conditions. For case (i) $q\alpha_1<p\alpha_2$, and $p\alpha_n<(1-p-q)\alpha_1$ or, case (ii) $q\alpha_1<p\alpha_2$, and $p\beta_n<(1-p-q)\beta_1$. One can verify that in both cases, R defines a nonempty triangular region in the (p,q) plane. For any $(p,q)\in R$, if any of the $3n$ inequalities in the majorization relationship $\lambda_{\psi\otimes\chi(p,q)} \prec \lambda_{\varphi\otimes\chi(p,q)}$ is an equality, then one of the two following cases must be true. Case (i) it is a *nonidentical* equality, i.e., the set of (p,q) that satisfies it defines a line in (p,q) plane, and hence comprises a measure zero set. Hence, the set of all points in R where there might be a nonidentical equality is of measure zero. Case (ii), it is a *benign* identity of the form $p\alpha_1+q\alpha_1+(1-p-q)\alpha_1 = p\beta_1+q\beta_1+(1-p-q)\beta_1$. ■

We now consider the case where the majorization inequalities contain both consecutive equalities and isolated ones. Let $\eta_{\psi,\varphi}$ be the size of the longest block of consecutive equalities in the majorization relationship of $|\psi\rangle$ and $|\varphi\rangle$. With the techniques that we have developed in this paper it is not difficult to show that if $\alpha_n\neq\beta_n$, then partial recovery of entanglement is always possible using auxiliary states of dimension $k=\eta_{\psi,\varphi}+2$. Thus, if $\alpha_n\neq\beta_n$, then *genuine* partial recovery is *always possible*, since $\alpha_n\neq\beta_n$, $\eta_{\psi,\varphi}\leqslant(n-3)$ and hence $k<n$.

In summary, we have shown that a nontrivial recovery is always possible except for the special case where $\alpha_n=\beta_n$, whether recovery is still possible for this special case is left as an open problem. There are many other open questions that might be of interest. For example, for a given pair of comparable states, one may ask what is the maximum entanglement that can be recovered. Similarly, can one recover more entanglement by increasing the dimension of the auxiliary entangled states? For example, we show that for almost all comparable states, 2×2 auxiliary states are sufficient to implement partial recovery, however, can one have more recovery of entanglement if the dimension of the auxiliary state is increased? We hope that the results of the present paper will lead to a better understanding of the subtleties involved in local entanglement manipulation in higher dimensions.

This work was sponsored in part by the Defense Advanced Research Projects Agency (DARPA) Project No. MDA972-99-1-0017, and in part by the U.S. Army Research Office/DARPA under Contract No. DAAD19-00-1-0172. The research of F.V. was supported also by a contract from the National Aeronautics and Space Administration. This work was also supported in part by the NSF under Grant No. EIA-0113440.

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